# The stress-energy tensor of a quark moving through a strongly-coupled $\mathcal{N}=4$ supersymmetric Yang-Mills plasma: comparing hydrodynamics and AdS/CFT

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The stress-energy tensor of a quark moving through a strongly coupled $\mathcal{N}=4$ supersymmetric Yang-Mills plasma is evaluated using gauge/string duality. The accuracy with which the resulting wake, in position space, is reproduced by hydrodynamics is examined. Remarkable agreement is found between hydrodynamics and the complete result down to distances less than $2/T$ away from the quark. In performing the gravitational analysis, we use a relatively simple formulation of the bulk to boundary problem in which the linearized Einstein field equations are fully decoupled. Our analysis easily generalizes to other sources in the bulk.



## I. INTRODUCTION

The discovery that the quark-gluon plasma produced in heavy ion collisions at RHIC behaves as a nearly ideal fluid [1, 2] has prompted much interest in understanding the dynamics of strongly coupled non-Abelian plasmas. Much recent theoretical work has explored the dynamics of maximally supersymmetric Yang-Mills ($\mathcal{N}=4$ SYM) plasma. (See, for example, Refs. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].) Thanks to gauge/string (or AdS/CFT) duality [18, 19], the properties of a strongly coupled $\mathcal{N}=4$ $SU(N_c)$ SYM plasma, in the limit of large $N_c$, are under much better theoretical control than is the case for a strongly coupled QCD plasma. Available evidence from the equation of state [18], screening lengths [12], and viscosity [6, 20], suggests that a strongly coupled, large-$N_c$, $\mathcal{N}=4$ SYM plasma mimics the properties of a real QCD plasma in the temperature range relevant to heavy ion collisions (roughly 1.5 to 4 times the deconfinement temperature) sufficiently well that one may view $\mathcal{N}=4$ SYM plasma as a useful model system for QCD.

Heavy quarks produced by hard processes during the early stages of a heavy ion collision may transfer much of their energy and momentum to the medium if they travel a sufficient distance through the fireball before potentially escaping (and hadronizing). Analysis of the distribution and correlations in produced jets can provide information about the dynamics of the plasma, including the rates of energy loss and momentum broadening of quarks traversing the plasma [21, 22, 23].

A heavy quark moving through a plasma will disturb the surrounding medium and its motion will, in turn, be influenced by the medium. As the quark moves, frictional forces will transfer energy and momentum from the quark to the plasma. A natural question to consider is where does the energy and momentum lost by the quark go? In other words, what is the form of the wake, as defined by the change in the expectation value of the stress-energy tensor,

$$\Delta T^{\mu\nu}(x) \equiv \langle T^{\mu\nu}(x)\rangle_{\text{with quark}} - \langle T^{\mu\nu}(x)\rangle_{\text{w/o quark}}. \quad (1.1)$$

For distances asymptotically far from the quark, one may address this question using hydrodynamic approximations. This approach was used in Ref. [24]. However, in this work the form of the effective sources for hydrodynamics were not fully specified.

Gauge/string duality allows one to compute many observables probing non-equilibrium dynamics of strongly coupled $\mathcal{N}=4$ supersymmetric Yang-Mills theory, including the rate of energy loss of a heavy quark moving through an SYM plasma [4]. (See also Refs. [3, 10, 14, 15, 25, 26, 27, 28] and references therein.) Recent studies have calculated the energy density [29, 30] and energy flux [31, 32] associated with a heavy quark moving through a strongly coupled SYM plasma. These studies have shown that qualitative features of the quark wake, such as the formation of a Mach cone for supersonic motion, match what is expected from hydrodynamics. However the comparison between hydrodynamics and the exact result for the stress-energy tensor has not yet been performed in a quantitative fashion. In particular, the range of validly of hydrodynamics has not been addressed quantitatively.

Hydrodynamics is valid only on length (or time) scales which are large compared to the mean free path (or time) of typical excitations in a fluid. In a weakly coupled relativistic plasma, the mean free path of quasiparticle excitations (quarks or gluons) is parametrically large compared to their de Broglie wavelength, $\ell_{\text{mfp}} \sim 1/(T\lambda^2 \ln \lambda^{-1}) \gg \ell_{\text{de Broglie}} \sim 1/T$ [33]. The 't Hooft coupling $\lambda \equiv g^2 N_c$ is the appropriate measure of the interaction strength. As the size of the coupling increases, this separation of scales shrinks, and hydrodynamics becomes valid on progressively shorter distance scales. When $\lambda \gtrsim 1$, a description in terms of weakly interacting quasiparticles is no longer valid. For an ultra-relativistic non-Abelian plasma in this regime, the minimum length scale $\ell_{\text{hydro}}$ on which hydrodynamics is valid must formally be of order $1/T$, as there is no other relevant scale. But whether, in practice, one needs $\ell_{\text{hydro}} \gtrsim 1/T$ or

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$\ell_{\rm hydro} \gtrsim 100/T$ in order for hydrodynamics to reproduce results of the full theory, to a given accuracy, cannot be determined without a quantitative comparison. Performing such a comparison is one goal of this paper.

To apply hydrodynamics to our situation of interest, in which a moving heavy quark is transferring energy and momentum to the plasma, one must first formulate appropriate effective sources to be used with the hydrodynamic equations. Just as in other applications of long distance effective field theories, this requires matching the effective theory to the underlying microscopic theory at the required level of precision. We show, on general grounds, that the appropriate effective sources for hydrodynamics can be simply expressed in terms of the drag force acting on the quark. Using gauge/string duality, we then compute the energy density and energy flux (or momentum density) associated with a heavy quark moving through a strongly coupled $\mathcal{N}=4$ SYM plasma, and compare the exact results to those obtained with hydrodynamics. Comparisons are made in both momentum space and position space. We find that the hydrodynamic approximation to the quark wake agrees with the exact result remarkably well even at distances less than $2/T$ away from the quark. Although our comparison applies to an $\mathcal{N}=4$ SYM plasma (in the limit of large $\lambda$ and large $N_{\rm c}$) the remarkably good performance of hydrodynamics in this context can only bolster the hope that it is sufficient to use hydrodynamics to model the transport of energy and momentum lost by high energy particles traversing a real quark-gluon plasma [34, 35] (despite the fact that gradients in the hydrodynamic variables can be disconcertingly large).

The outline of the remainder of this paper is as follows. In Section II we introduce various definitions and conventions that we will use throughout the analysis. In Section III we discuss the hydrodynamic description of perturbations in the SYM stress-energy tensor. This includes the formulation of effective sources for the hydrodynamic equations and the separation of the energy flux into contributions from sound and diffusion modes. In Section IV we turn to the dual gravitational formulation. Gauge/string duality maps the problem of computing the perturbation in the stress-energy due to the quark, $\Delta T^{\mu\nu}(x)$, into the problem of computing the perturbation to the geometry caused by an open string (dual to the heavy quark) moving through a five dimensional anti-de Sitter/Schwarzschild spacetime. We introduce a convenient set of gauge invariant variables which encode the metric perturbation, show how their use allows one to decouple completely the linearized Einstein equations, and then, by analyzing the on-shell gravitational action, show how to reconstruct the expectation value of the SYM stress-energy tensor using our chosen gauge invariant variables. We also describe our technique for solving, numerically, the ordinary differential equations satisfied by the gauge invariant variables, which is based on the construction of appropriate Green's functions from numerically computed homogeneous solutions. We present, in Section V, the results of the computation of the perturbation in the energy density and energy flux, in position space, due to a moving quark, at several different velocities. We also examine the small momentum asymptotics and compare results, in several ways, with hydrodynamics. Section VI discusses the interpretation of our results. We conclude in Section VII. Two appendices contain details of the analysis of the boundary action, and the extraction of small momentum asymptotics.

We have endeavored to make the presentation relatively self-contained. Readers interested in the results but not the details of the gravitational calculation should feel free to skip over Section IV.

## II. DEFINITIONS AND CONVENTIONS

We use the Minkowski space metric $\eta_{\mu\nu} \equiv \mathrm{diag}(-1,1,1,1)$. Five dimensional AdS coordinates will be denoted by $X_M$ while four dimensional Minkowski coordinates will be denoted by $x_\mu$. Upper case Latin indices $M, N, P, \ldots$ run over $5d$ AdS coordinates, while Greek indices run over the $4d$ Minkowski space coordinates. We choose coordinates such that the metric of the AdS-Schwarzschild (AdS-BH) geometry is

$$ds^2 = \frac{L^2}{u^2}\left[-f(u)\,dt^2 + d\boldsymbol{x}^2 + \frac{du^2}{f(u)}\right], \qquad (2.1)$$

where $f(u) \equiv 1-(u/u_h)^4$ and $L$ is the AdS curvature radius. The coordinate $u$ is an inverse radial coordinate; the boundary of the AdS-BH spacetime is at $u=0$ and the event horizon is located at $u=u_h$, with $T \equiv (\pi u_h)^{-1}$ the temperature of the SYM plasma.

When introducing a Fourier transform over the $4d$ Minkowski space coordinates, we will often decompose vectors and tensors in terms of an orthonormal basis of polarization vectors $\{\hat{\boldsymbol{q}},\, \hat{\boldsymbol{\epsilon}}_1,\, \hat{\boldsymbol{\epsilon}}_2\}$. The quark's velocity $\boldsymbol{v}$ defines a preferred direction so a natural choice of polarization vectors is

$$\hat{\boldsymbol{\epsilon}}_1 = \frac{q}{q_\perp}\,\hat{\boldsymbol{q}}\times(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{q}})\,, \quad \hat{\boldsymbol{\epsilon}}_2 = \frac{q}{q_\perp}\,\hat{\boldsymbol{v}}\times\hat{\boldsymbol{q}}\,, \qquad (2.2)$$

where $q \equiv |\boldsymbol{q}|$ and $q_\perp = |\boldsymbol{q} - (\hat{\boldsymbol{v}}\cdot\boldsymbol{q})\,\hat{\boldsymbol{v}}|$ is the magnitude of the component of $\boldsymbol{q}$ orthogonal to $\boldsymbol{v}$. Lower case Latin indices $a, b, \cdots = 1,2$ will be used to refer to the *transverse* spatial components of vectors and tensors. We will decompose the Fourier transform $A^M(\omega,\boldsymbol{q})$ of any given vector field as follows,

$$A^0 \equiv \mathcal{A}^0, \qquad (2.3a)$$

$$A^5 \equiv \mathcal{A}^5, \qquad (2.3b)$$

$$A^i \equiv \hat{q}^i\mathcal{A}^q + \epsilon^i_a\,\mathcal{A}^a\,, \qquad (2.3c)$$

with a sum over repeated indices implied. The components $\mathcal{A}^M \equiv \{\mathcal{A}^0, \mathcal{A}^a, \mathcal{A}^q, \mathcal{A}^5\}$ will be referred to as the components of the vector field $A$ in the polarization

frame. Higher rank tensors will be represented by polarization frame components in the analogous fashion.

It will also prove convenient to use the notation

$$\vec{Z}_1 \equiv Z_1^a \, \hat{\boldsymbol{\epsilon}}_a \,, \qquad \overleftrightarrow{Z}_2 \equiv Z_2^{ab} \, \hat{\boldsymbol{\epsilon}}_a \otimes \hat{\boldsymbol{\epsilon}}_b \,, \tag{2.4}$$

for quantities which transform with helicity one or two under SO(2) rotations about the $\hat{\boldsymbol{q}}$ axis.

In four dimensions, any symmetric tensor field $T^{\mu\nu}$ which satisfies a conservation equation,

$$\partial_\mu T^{\mu\nu} = V^\nu \,, \tag{2.5}$$

and the trace condition,

$$T^\mu_{\ \mu} = \beta \,, \tag{2.6}$$

for some given $V^\mu$ and $\beta$, contains only five independent degrees of freedom. A convenient representation of these independent degrees of freedom is provided by the following helicity zero, one, and two components of the Fourier transformed tensor:

$$\mathcal{T}_0 \equiv \mathcal{T}^{00}, \tag{2.7a}$$

$$\vec{\mathcal{T}}_1 \equiv \mathcal{T}^{0a} \, \hat{\epsilon}_a, \tag{2.7b}$$

$$\overleftrightarrow{\mathcal{T}}_2 \equiv \left(\mathcal{T}^{ab} - \tfrac{1}{2} \, \mathcal{T}^{cc} \, \delta^{ab}\right) \hat{\epsilon}_a \otimes \hat{\epsilon}_b \,, \tag{2.7c}$$

where $\mathcal{T}^{\mu\nu}$ are the components of $T^{\mu\nu}$ in the polarization frame. The reconstruction of the original tensor components $T^{\mu\nu}$ is given by

$$T^{00} = \mathcal{T}_0 \,, \tag{2.8a}$$

$$T^{0i} = \hat{\epsilon}^i_a \, \mathcal{T}^a_1 + \hat{q}^i \, \mathcal{T}^{0q} \,, \tag{2.8b}$$

$$T^{ij} = \hat{\epsilon}^i_a \hat{\epsilon}^j_b \, \mathcal{T}^{ab}_2 + \hat{q}^{(i} \hat{\epsilon}^{j)}_a \, \mathcal{T}^{aq} + \tfrac{3}{2} \left(\hat{q}^i \hat{q}^j - \tfrac{1}{3} \, \delta^{ij}\right) \mathcal{T}^{qq} + \tfrac{1}{2} \left(\delta^{ij} - \hat{q}^i \hat{q}^j\right) \left(\beta + \mathcal{T}^{00}\right) , \tag{2.8c}$$

where $v^{(i} u^{j)} \equiv v^i u^j + v^j u^i$ and

$$\mathcal{T}^{0q} \equiv \left(\omega \mathcal{T}_0 - i \mathcal{V}^0\right) / q \,, \tag{2.9a}$$

$$\mathcal{T}^{aq} \equiv \left(\omega \mathcal{T}^a_1 - i \mathcal{V}^a\right) / q \,, \tag{2.9b}$$

$$\mathcal{T}^{qq} \equiv \left(\omega \mathcal{T}^{0q} - i \mathcal{V}^q\right) / q = \left[\omega^2 \mathcal{T}_0 - i(\omega \mathcal{V}^0 + q \mathcal{V}^q)\right] / q^2 \,, \tag{2.9c}$$

with $\mathcal{V}^\mu$ the components of $V^\mu$ in the polarization frame. We will refer to the quantities $\{\mathcal{T}_s\}$ as *helicity variables* and the representation of $T^{\mu\nu}$ in terms of $\mathcal{T}_s$ as the *helicity decomposition* of $T^{\mu\nu}$. We emphasize that the helicity decomposition is complete only when the helicity variables $\mathcal{T}_s$ *and* both $\beta$ and $V^\mu$ are known.

## III. HYDRODYNAMIC DESCRIPTION

We consider a fundamental representation quark of mass $M$ moving with constant velocity $\boldsymbol{v}$ through an $\mathcal{N}=4$ SYM plasma at temperature $T$. We assume that both the quark mass $M$, and its kinetic energy, are large compared to $T$.[1]

We assume that the quark has been moving at the velocity $\boldsymbol{v}$ for an arbitrarily long time $\Delta t \to \infty$. Due to its interaction with the SYM gauge (and scalar) fields, the quark will perturb the surrounding plasma and the plasma will exert a friction force, or drag, on the quark. The drag force on the quark is minus the rate at which momentum is transferred from the quark to the plasma. In the absence of any external force, the drag exerted by the plasma would cause the quark to lose momentum and slow down. To maintain a constant velocity, energy and momentum must be supplied to the quark via an external force which exactly counterbalances the plasma drag (at a given terminal velocity). This is naturally accomplished by turning on a constant background $U(1)$ electric field which couples to the fundamental representation quark but not to the adjoint representation SYM degrees of freedom.

The microscopic energy-momentum conservation equation takes the form

$$\partial_\mu T^{\mu\nu}(x) = F^\nu(x) \,, \tag{3.1}$$

where $T^{\mu\nu}$ is the stress-energy tensor for the system (not including the background $U(1)$ electric field), and $F^\nu$ is the external force density, or minus the drag force density, acting on the quark. In the limit of large quark mass, the quark can be arbitrarily well-localized and the force density will have point support. In this regime we may write

$$F^\mu(t, \boldsymbol{x}) = f^\mu \, \delta^3(\boldsymbol{x} - \boldsymbol{v}t) \,, \tag{3.2}$$

where $f^\mu \equiv dp^\mu_{\text{quark}}/dt$ is the external force (or minus the drag force) acting on the quark, and coordinates are chosen so that the quark is at the origin at time $t = 0$.[2] For strongly coupled SYM, the magnitude of the drag force has been evaluated (using gauge/string duality) and one finds [3, 4]

$$\boldsymbol{f} = \frac{\pi}{2} \sqrt{\lambda} T^2 \frac{\boldsymbol{v}}{\sqrt{1 - v^2}} \,, \tag{3.3a}$$

$$f^0 = \boldsymbol{f} \cdot \boldsymbol{v} \,. \tag{3.3b}$$

For distances $d \equiv |\boldsymbol{x} - \boldsymbol{v}t|$ of order $1/T$ or less, gradients in the stress-energy tensor are large, non-hydrodynamic

[1] Having kinetic energy large compared to $T$ implies that Brownian motion of the quark, induced by thermal fluctuations in the plasma, may be neglected [4]. Our analysis in the gravitational dual will require a stronger condition on the quark mass, $M \gg \sqrt{\lambda}\, T$, where $\lambda$ is the (large) 't Hooft coupling. Strongly-coupled SYM with massive fundamental hypermultiplets has deeply bound mesons whose masses scale as the quark mass $M$ divided by $\sqrt{\lambda}$, so the condition $M \gg \sqrt{\lambda}\, T$ implies that both mesons and quarks are heavy compared to $T$.

[2] Note that $f^\mu$ is the four-momentum transfer per unit coordinate time, not proper time; the covariant four-force $dp^\mu_{\text{quark}}/d\tau$ equals $f^\mu/\sqrt{1 - v^2}$.

degrees of freedom are important, and the complete microscopic theory is needed to compute the transport of energy and momentum. But as the disturbance in the plasma, induced by the quark, propagates out to larger distances, dissipative effects will decrease the size of gradients. At sufficiently large distances from the quark, the transport of energy and momentum can be described by neutral fluid hydrodynamics. This is the appropriate effective theory for the long wavelength, slowly relaxing degrees of freedom in a non-Abelian plasma.[3] The relevant hydrodynamic variables are the locally conserved energy-momentum densities $T^{0\mu}(x)$. In particular, as we next discuss, long wavelength perturbations in the spatial stress tensor can be expressed entirely in terms of these conserved densities.

Instead of working directly with the conserved densities $T^{0\mu}(x)$, it is conventional, and convenient, to introduce the proper energy density $\epsilon(x)$ and the fluid four-velocity $u^\mu(x)$. The fluid four-velocity is defined as the velocity of a local reference frame in which the spatial momentum density vanishes, and the proper energy density is the energy density in this local fluid rest frame.

We will denote with a bar the components of the stress-energy tensor in the local fluid rest frame (at the location $x$), If the fluid were in perfect equilibrium, with a globally-defined rest frame, then in that rest frame the stress-energy tensor would have the form,

$$\overline{T}^{00}_{\rm eq} \equiv \epsilon \,, \tag{3.4a}$$
$$\overline{T}^{0i}_{\rm eq} \equiv 0 \,, \tag{3.4b}$$
$$\overline{T}^{ij}_{\rm eq} \equiv p\,\delta_{ij} \,, \tag{3.4c}$$

with the pressure $p$ and energy density $\epsilon$ related by the equilibrium equation of state of the fluid, $p = p(\epsilon)$. When the fluid is not in perfect equilibrium, the definition of the local fluid rest frame allows one to write

$$\overline{T}^{00}(x) \equiv \epsilon(x) \,, \tag{3.5a}$$
$$\overline{T}^{0i}(x) \equiv 0 \,, \tag{3.5b}$$
$$\overline{T}^{ij}(x) \equiv p(x)\,\delta_{ij} + \tau_{ij}(x) \,, \tag{3.5c}$$

where $p(x) \equiv p(\epsilon(x))$ is the equilibrium value of the pressure which corresponds (via the equation of state) to the local energy density $\epsilon(x)$, and all non-equilibrium effects are contained in the dissipative contribution to the stress tensor, $\tau_{ij}$.

For sufficiently long wavelength variations in the energy and momentum density (or proper energy density and fluid velocity), the dissipative contribution to the stress may be expanded in terms of spatial gradients of the hydrodynamic variables. This, by definition, is the hydrodynamic regime.

It is straightforward to construct the gradient expansion of $\tau_{ij}$. However this exercise is simplified when (i) the underlying theory is conformal, and (ii) the variations in the energy and momentum density are parametrically small compared to their equilibrium value. In conformal theories, such as $\mathcal{N}=4$ SYM, the stress-energy tensor is traceless. Consequently, $\epsilon = 3p$ and $\tau_{ii} = 0$. If the variations in energy and momentum density are small, then one may also expand in powers of departures from equilibrium. This is the case in our application involving a single fundamental representation quark interacting with a large $N_{\rm c}$ SYM plasma. In the large $N_{\rm c}$ limit, the equilibrium stress-energy tensor of the plasma is $\mathcal{O}(N_{\rm c}^2)$ while the perturbations in $T^{\mu\nu}$ due to the addition of a single quark are $\mathcal{O}(N_{\rm c}^0)$. Consequently, both the fractional change in the energy density and the fluid velocity induced by the moving quark are of order $1/N_{\rm c}^2$. It follows that the equations determining perturbations in the stress-energy tensor must be linear in the large $N_{\rm c}$ limit. More specifically, nonlinear terms in the hydrodynamic equations of motion are suppressed by powers of $N_{\rm c}$. For later convenience, let

$$\mathcal{E}(x) \equiv \epsilon(x) - \epsilon_{\rm eq} \,, \tag{3.6a}$$

and

$$\mathcal{P}(x) \equiv \frac{\partial p}{\partial \epsilon}\,\mathcal{E}(x) = \tfrac{1}{3}\,\mathcal{E}(x) \,, \tag{3.6b}$$

denote, respectively, the deviation of the energy density from its equilibrium value and the associated deviation in the pressure. Linearity of the hydrodynamic equations of motion implies that only terms linear in $\boldsymbol{u}$ and $\mathcal{E}$ (and their derivatives) will be needed in the gradient expansion of the stress tensor. With this and the vanishing trace condition in mind, the gradient expansion of the dissipative stress $\tau_{ij}$ has the form

$$\begin{aligned}\tau_{ij} = &-\eta\left(\nabla_i u_j + \nabla_j u_i - \tfrac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right)\\ &+ \Theta\left(\nabla_i\nabla_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\mathcal{E} + \cdots .\end{aligned} \tag{3.7}$$

The coefficient $\eta$ is the shear viscosity, while the coefficient $\Theta$ characterizes the first correction to viscous hydrodynamics.

The stress-energy tensor in the local fluid rest frame is related to the stress-energy tensor in the lab frame by the local boost (in block form)

$$\Lambda(x) = \begin{pmatrix} 1 & \boldsymbol{u}(x)^{\rm T} \\ \boldsymbol{u}(x) & \mathbf{1} \end{pmatrix} + \mathcal{O}(\boldsymbol{u}(x)^2). \tag{3.8}$$

Hence, the stress-energy tensor in the lab frame is

$$T^{\mu\nu}_{\rm hydro}(x) = T^{\mu\nu}_{\rm eq} + \Delta T^{\mu\nu}_{\rm hydro}(x) \,, \tag{3.9}$$

where

$$T^{\mu\nu}_{\rm eq} = {\rm diag}(\epsilon_{\rm eq}, p_{\rm eq}, p_{\rm eq}, p_{\rm eq}) \,, \tag{3.10}$$

[3] In Abelian plasmas, the appropriate effective theory is magnetohydrodynamics as the magnetic field can have an arbitrarily long correlation length. But in non-Abelian plasmas, even static magnetic fluctuations of the gauge field develop develop a finite correlation length.

and

$$\Delta T^{00}_{\rm hydro} = \mathcal{E}\,, \tag{3.11a}$$

$$\Delta T^{0i}_{\rm hydro} = (\epsilon_{\rm eq} + p_{\rm eq})\, u_i\,, \tag{3.11b}$$

$$\Delta T^{ij}_{\rm hydro} = \mathcal{P}\,\delta_{ij} - \eta\left(\nabla_i u_j + \nabla_j u_i - \tfrac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right) + \Theta\left(\nabla_i\nabla_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\mathcal{E} + \cdots\,, \tag{3.11c}$$

(neglecting nonlinear corrections subleading in $1/N_c^2$). Further derivative corrections to the above formulas only enter in Eq. (3.11c), which is the constitutive relation expressing the spatial stress tensor in terms of the hydrodynamic variables.

### A. Equations of Motion and Effective Source

The equations of motion for the hydrodynamic fluctuations follow from applying the energy-momentum conservation equation (3.1) to $T^{\mu\nu}_{\rm hydro}$. However, the hydrodynamic decomposition of the stress tensor is not valid in the near zone close to where the source $F^\nu$ is non-vanishing. In this region, gradients become large and truncating the gradient expansion in Eq. (3.7) is unjustified, as is the neglect of all non-hydrodynamic degrees of freedom. However, in the spirit of effective field theory, one may formulate effective sources for the hydrodynamic equations, which encapsulate the dynamics in the near zone. Doing so, we write the energy conservation relation for the hydrodynamic fluctuations as

$$\partial_\mu T^{\mu\nu}_{\rm hydro}(x) = J^\nu(x)\,, \tag{3.12}$$

where $J^\nu$ is the effective source. Because hydrodynamics is a long distance effective theory, the effective source can be regarded as having point support at the location of the quark. That is, $J^\mu$ may be expanded in terms of delta functions and their derivatives.[4] To leading order in gradients we have

$$J^\mu(t,\boldsymbol{x}) = j^\mu_{(0)}\,\delta^3(\boldsymbol{x}-\boldsymbol{v}t) + \cdots\,. \tag{3.13}$$

The vector $j^\mu_{(0)}$ can in principle depend on time. However, during the (assumed large) interval of time in which the quark is moving at constant velocity, $j^\mu_{(0)}$ will be time independent.

The effective source $J^\mu(x)$ can be determined by calculating the long wavelength limit of the stress-energy tensor via the full quantum theory, and then matching its form to hydrodynamics. We do this in Section V and find, in particular, that the coefficient $j^\mu_{(0)}$ of the leading term in the expansion (3.13) is simply equal the rate at which the quark transfers four-momentum to the plasma, so that $j^\mu_{(0)}$ coincides with the external force (3.3). This value for the leading term in the effective source can also be deduced with the following simple argument, which nicely highlights the fact that this relation between the effective source $j^\mu_{(0)}$ and the microscopic drag force emerges even though hydrodynamics is not applicable in a near zone surrounding the quark.

Assume that the electric field pulling the quark is turned on at time $t_i$, and turned off at some later time $t_f$. The total four-momentum transfered from the quark to the plasma is

$$\Delta p^\mu = \int_{t_i}^{t_f} dt\; f^\mu(\boldsymbol{v}(t))\,, \tag{3.14}$$

where $f^\mu(\boldsymbol{v})$ is the external force (3.3) (or minus the drag force) for velocity $\boldsymbol{v}$. For times $t$ much later than $t_f$, all of the four-momentum transfered to the plasma will have been transported to distances far from the quark. (As discussed below, since the relevant dynamics is diffusive, the characteristic distance $d$ is proportional to $\sqrt{t-t_f}$ and diverges for large $t$.) The resulting four-momentum stored in the far zone is computable using hydrodynamics and, for late times $t \gg t_f$, this far-zone four-momentum must match the four-momentum (3.14) lost by the quark.

In the limit in which the quark has been moving at constant velocity $\boldsymbol{v}$ for an arbitrarily long period of time, the total four-momentum transfered to the plasma (neglecting irrelevant endpoint corrections sensitive to how the electric field is ramped up and down) is just

$$\Delta p^\mu \approx (t_f - t_i)\, f^\mu(\boldsymbol{v})\,. \tag{3.15}$$

In the hydrodynamic description, the total momentum transfered to the far zone at time $t \gg t_f$ is given by

$$\Delta p^\mu_{\rm hydro}(t) = \int d^3x\; \Delta T^{0\mu}_{\rm hydro}(t,\boldsymbol{x})\,. \tag{3.16}$$

Rewriting this as

$$\Delta p^\mu_{\rm hydro}(t) = \int_{-\infty}^{t} dt'\,\frac{\partial}{\partial t'}\int d^3x\; \Delta T^{0\mu}_{\rm hydro}(t',\boldsymbol{x})\,, \tag{3.17}$$

and then using the effective energy-momentum conservation relation (3.12), yields

$$\Delta p^\mu_{\rm hydro}(t) = \int_{-\infty}^{t} dt' \int d^3x\left[-\nabla_i\,\Delta T^{i\mu}_{\rm hydro} + J^\mu\right]. \tag{3.18}$$

The first term in the integral can be converted to a surface integral over the sphere at spatial infinity. This surface term vanishes by causality — if the quark has only been moving for a finite period of time, then perturbations in the stress-energy tensor must vanish at spatial

[4] In other words, the spatial Fourier transform of $J^\nu$ may be expanded in a Taylor series in the wavevector $\boldsymbol{q}$. This is completely analogous to the situation in electromagnetism where, given localized charge and current densities, one may similarly represent the charge and current densities as a sum of delta functions and their derivatives, and this directly leads to the multipole expansion for the electric and magnetic fields in the far zone.

infinity. In the second term, the spatial integral of $J^\mu$ just gives the leading term in the expansion (3.13) of the effective source, $\int d^3x\, J^\mu = j^\mu_{(0)}$ and, as noted above, $j^\mu_{(0)}$ will be time independent during the long interval during which the quark moves with constant velocity. Consequently, for $t \gg t_f$, we have

$$\Delta p^\mu_{\rm hydro}(t) \approx (t_f - t_i)\, j^\mu_{(0)}\,. \tag{3.19}$$

Demanding that this reproduce the microscopic result (3.15) shows that

$$j^\mu_{(0)} = f^\mu\,, \tag{3.20}$$

as asserted above.

Higher order gradient corrections to $J^\mu$, which we evaluate in Section V, lead to corrections to the stress-energy tensor which are suppressed by additional inverse powers of $|\boldsymbol{x}|$ in the far zone. We note however, that in the large $N_{\rm c}$ limit

$$J^0(t,\boldsymbol{x}) \equiv F^0(t,\boldsymbol{x}) = \boldsymbol{f}\cdot\boldsymbol{v}\;\delta^3(\boldsymbol{x}-\boldsymbol{v}t)\,, \tag{3.21}$$

with no additional derivative corrections. This follows from the fact that the linearized hydrodynamic expressions for the conserved densities $\Delta T^{0\nu}_{\rm hydro}$, as given in Eqs. (3.11a) and (3.11b), do not receive any derivative corrections (and all non-linear terms are suppressed by additional factors of $1/N_{\rm c}$). The form of the gradient corrections to $\boldsymbol{J}$ are also constrained. These constraints are conveniently formulated in Fourier space. If the duration of time $\Delta t$ in which the quark has been moving at constant velocity $\boldsymbol{v}$ is sent to infinity, then $\boldsymbol{J}(\omega,\boldsymbol{q})$ will be proportional to $2\pi\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})$. Furthermore, $\boldsymbol{J}(\omega,\boldsymbol{q})$ can only depend on the vectors $\boldsymbol{v}$, $\boldsymbol{q}$, which implies that $\boldsymbol{J}(\omega,\boldsymbol{q})$ must lie in the plane spanned by these vectors.[5] We therefore can write

$$\boldsymbol{J}(\omega,\boldsymbol{q}) = \big[\boldsymbol{v}\,\phi_v(\omega,q^2) + i\boldsymbol{q}\,\phi_q(\omega,q^2)\big]\,2\pi\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\,. \tag{3.22}$$

The functions $\phi_v(\omega,q^2)$ and $\phi_q(\omega,q^2)$ must be analytic in both $\omega$ and $q^2$, in a neighborhood of $\omega=\boldsymbol{q}=0$, so that $\boldsymbol{J}(t,\boldsymbol{x})$ has an expansion in terms of derivatives of delta functions with point support at the location of the quark. The condition (3.20) implies that $\phi_v(0,0) = \boldsymbol{f}\cdot\boldsymbol{v}/v^2$.

### B. Sound and Diffusion Modes

To compare with the gravitational results presented in Section IV it will be advantageous to perform a spacetime Fourier transform and express the stress-energy tensor in terms of the helicity decomposition discussed in Section II. In particular, $\Delta T^{\mu\nu}_{\rm hydro}$ can be reconstructed from the helicity variables $\Delta\mathcal{T}_0^{\rm hydro}$, $\Delta\vec{\mathcal{T}}_1^{\rm hydro}$ and $\Delta\overleftrightarrow{\mathcal{T}}_2^{\rm hydro}$, as summarized in Eqs. (2.8)–(2.9) with $V^\mu = J^\mu$ and $\beta=0$. The ansatz given in Eq. (3.11) and the effective conservation relation (3.12) determine the functional form of the helicity variables. Let $\boldsymbol{J}_{\rm L}$ and $\boldsymbol{J}_{\rm T}$ be the transverse and longitudinal components of $\boldsymbol{J}$. Ignoring the second order $\Theta$ term and higher order derivative corrections in Eq. (3.11c), one finds that the helicity variables are given by

$$\Delta\mathcal{T}_0^{\rm hydro} = \frac{\rho}{\omega^2 - c_{\rm s}^2\,q^2 + i\gamma q^2\omega}\,, \tag{3.23a}$$

$$\Delta\vec{\mathcal{T}}_1^{\rm hydro} = \frac{\boldsymbol{J}_{\rm T}}{-i\omega + Dq^2}\,, \tag{3.23b}$$

$$\Delta\overleftrightarrow{\mathcal{T}}_2^{\rm hydro} = 0\,, \tag{3.23c}$$

where

$$\rho \equiv i\boldsymbol{q}\cdot\boldsymbol{J} + i\omega\,J^0 - \gamma\,q^2 J^0\,. \tag{3.24}$$

Here $c_{\rm s}^2 = \partial\epsilon/\partial p$ is the speed of sound, $\gamma = 4\eta/3(\epsilon+p)$ is the sound attenuation constant, and $D = \eta/(\epsilon+p)$ is the transverse momentum diffusion constant. For strongly coupled SYM [5, 36]

$$c_{\rm s}^2 = 1/3\,, \tag{3.25a}$$

$$\gamma = 1/(3\pi T)\,, \tag{3.25b}$$

$$D = 1/(4\pi T)\,. \tag{3.25c}$$

The lack of any helicity two component, Eq. (3.23c), remains true (in the large $N_{\rm c}$ limit) even when higher order gradient corrections are included. This is a consequence of the fact that $\Delta T^{ij}_{\rm hydro}$ is linear in the perturbations $\mathcal{E}$ and $u^i$. With this constraint, regardless of the number of spatial derivatives, one cannot construct a nonzero transverse traceless component of the spatial stress.

It is straightforward to reconstruct the perturbation in the energy density $\mathcal{E}$ and energy flux $S_i \equiv \Delta T^{0i}_{\rm hydro}$ from the helicity variables and the effective energy-momentum conservation relation (3.12). The energy density and energy flux, in spacetime, are given by

$$\mathcal{E}(t,\boldsymbol{x}) = \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\;\frac{\rho}{\omega^2 - c_{\rm s}^2\,q^2 + i\gamma\,q^2\omega}\;e^{iQ\cdot x}, \tag{3.26}$$

$$\boldsymbol{S}_{\rm L}(t,\boldsymbol{x}) = \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\;\frac{c_{\rm s}^2\,i\boldsymbol{q}J^0 + i\omega\boldsymbol{J}_L}{\omega^2 - c_{\rm s}^2\,q^2 + i\gamma\,q^2\omega}\;e^{iQ\cdot x}, \tag{3.27}$$

$$\boldsymbol{S}_{\rm T}(t,\boldsymbol{x}) = \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\;\frac{-\boldsymbol{J}_{\rm T}}{i\omega - D\,q^2}\;e^{iQ\cdot x}\,, \tag{3.28}$$

with $\boldsymbol{S}(t,\boldsymbol{x}) \equiv \boldsymbol{S}_{\rm L}(t,\boldsymbol{x}) + \boldsymbol{S}_{\rm T}(t,\boldsymbol{x})$ and the four-vector $Q\equiv(\omega,\boldsymbol{q})$. These expressions show that the energy density and the longitudinal component $\boldsymbol{S}_{\rm L}$ of the energy

[5] In principle one can consider an additional component of $\boldsymbol{J}$ proportional to $\boldsymbol{v}\times\boldsymbol{q}$. Since this is a pseudovector, its coefficient function must be a pseudoscalar. But there is no pseudoscalar that can be constructed out of $\boldsymbol{v}$ and $\boldsymbol{q}$ alone, so this term is not allowed.

flux satisfy a diffusive wave equation, while the transverse component $\boldsymbol{S}_{\rm T}$ of the energy flux satisfies a diffusion equation. The energy density obeys the inhomogeneous wave equation

$$\left(-\partial_0^2 + c_{\rm s}^2\,\nabla^2 + \gamma\nabla^2\partial_0\right)\mathcal{E} = \rho\,, \tag{3.29}$$

which describes damped sound waves, traveling at speed $c_{\rm s} = 1/\sqrt{3}$, with a source $\rho$ given by Eq. (3.24).

It is instructive to separate the energy flux into sound and diffusion modes. Since the longitudinal part of the flux $\boldsymbol{S}_{\rm L}$ satisfies a wave equation while the transverse part $\boldsymbol{S}_{\rm T}$ satisfies a diffusion equation, one would naturally expect that $\boldsymbol{S}_{\rm L}$ should be identified with the energy flux carried by sound waves, while $\boldsymbol{S}_{\rm T}$ characterizes diffusive energy flux. But, in a steady-state situation where the quark has been moving for an arbitrarily long time, things are not so simple. The effective source $\boldsymbol{J}(\omega,\boldsymbol{q})$, Eq. (3.22), equals $\delta(\omega - \boldsymbol{v}\cdot\boldsymbol{q})$, times a function which is regular as $\omega$ and $\boldsymbol{q}\to 0$. But its longitudinal projection has a directional singularity at $q=0$,

$$\boldsymbol{J}_{\rm L} = \frac{\boldsymbol{q}\,(\boldsymbol{q}\cdot\boldsymbol{J})}{q^2} = \boldsymbol{q}\left[\frac{\boldsymbol{v}\cdot\boldsymbol{q}}{q^2}\,\phi_v(\omega,q^2) + \cdots\right] 2\pi\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,, \tag{3.30}$$

with $\cdots$ denoting terms regular as $\boldsymbol{q}\to 0$. Therefore, the Fourier transform of $\boldsymbol{J}_{\rm L}$ is not localized at the quark, but rather falls off in space like the inverse cube of the distance away from the quark. As a result, the longitudinal energy flux $\boldsymbol{S}_{\rm L}$, defined by Eq. (3.27), contains the term[6]

$$\boldsymbol{S}_{\rm nonlocal} = \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{i\boldsymbol{q}}{q^2}\,\phi_v(\omega,q^2)\,2\pi\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,e^{iQ\cdot x}\,. \tag{3.31}$$

This gives a Coulomb-field-like flux at large distance,

$$\boldsymbol{S}_{\rm nonlocal}(t,\boldsymbol{x}) \sim \nabla\left(\frac{1}{4\pi v|\boldsymbol{x}-\boldsymbol{v}t|}\right)\frac{\boldsymbol{f}\cdot\boldsymbol{v}}{v^2}\,. \tag{3.32}$$

This is co-moving with the quark and does not propagate like a wave. It makes little sense to regard this as a contribution to the energy flux carried by sound waves. (Note, in particular, that $\boldsymbol{S}_{\rm nonlocal}$ is non-vanishing in front of the quark even when the quark is moving supersonically.)

Since the transverse part of the effective source, $\boldsymbol{J}_{\rm T}$ equals $\boldsymbol{J}$ (which is regular as $\boldsymbol{q}\to 0$) minus $\boldsymbol{J}_{\rm L}$, a completely parallel argument shows that the transverse flux $S_{\rm T}$, as defined in Eq. (3.28), contains a non-diffusing contribution which is precisely $-\boldsymbol{S}_{\rm nonlocal}$.

The presence of these equal and opposite Coulomb-like contributions naturally suggests that the energy flux which is properly associated with sound and diffusion can be obtained by suitably adding and subtracting this term. To this end, we write

$$\boldsymbol{S} = \boldsymbol{S}_{\rm sound} + \boldsymbol{S}_{\rm diffusion}\,, \tag{3.33}$$

with

$$\boldsymbol{S}_{\rm sound} \equiv \boldsymbol{S}_{\rm L} - \boldsymbol{S}_{\rm nonlocal}\,, \tag{3.34a}$$

$$\boldsymbol{S}_{\rm diffusion} \equiv \boldsymbol{S}_{\rm T} + \boldsymbol{S}_{\rm nonlocal}\,. \tag{3.34b}$$

Using Eq. (3.27), it is straightforward to see that $\boldsymbol{S}_{\rm sound}$ satisfies the damped wave equation

$$\left(-\partial_0^2 + c_{\rm s}^2\,\nabla^2 + \gamma\nabla^2\partial_0\right)\boldsymbol{S}_{\rm sound} = \boldsymbol{J}_{\rm sound} \tag{3.35}$$

with a source

$$\begin{aligned}\boldsymbol{J}_{\rm sound}(t,\boldsymbol{x}) \equiv \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,i\boldsymbol{q}\,(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\,e^{iQ\cdot x}\\ \times\left[c_{\rm s}^2 f^0 + (c_{\rm s}^2{-}i\gamma\omega)\phi_v(\omega,q^2) + i\omega\,\phi_q(\omega,q^2)\right].\end{aligned} \tag{3.36}$$

Unlike $\boldsymbol{J}_{\rm L}(\omega,\boldsymbol{q})$, the integrand defining the modified source $\boldsymbol{J}_{\rm sound}$ is now regular as $\boldsymbol{q}\to 0$. Hence $\boldsymbol{J}_{\rm sound}$ is localized at the quark (it may be expanded in terms of delta functions and their derivatives). Similarly, using Eq. (3.28), one finds that $\boldsymbol{S}_{\rm diffusion}$ satisfies the inhomogeneous diffusion equation

$$\left(\partial_0 - D\nabla^2\right)\boldsymbol{S}_{\rm diffusion} = \boldsymbol{J}_{\rm diffusion}\,, \tag{3.37}$$

with the source

$$\begin{aligned}\boldsymbol{J}_{\rm diffusion} \equiv \int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,e^{iQ\cdot x}\\ \times\,(\boldsymbol{v}+i\boldsymbol{q}D)\,\phi_v(\omega,q^2)\,,\end{aligned} \tag{3.38}$$

which is also localized at the quark. Because the redefined sources $\boldsymbol{J}_{\rm sound}$ and $\boldsymbol{J}_{\rm diffusion}$ are localized, the corresponding energy fluxes, $\boldsymbol{S}_{\rm sound}$ and $\boldsymbol{S}_{\rm diffusion}$, may be regarded as a sensible decomposition of the flux into contributions from propagating and diffusing degrees of freedom.[7]

[6] To see this, use the frequency delta function to rewrite the factor of $\omega$ multiplying $\boldsymbol{J}_{\rm L}$ in Eq. (3.27) times the $(\boldsymbol{v}\cdot\boldsymbol{q})$ factor in the singular term (3.30) as $\omega^2 = (\omega^2 - c_{\rm s}^2 q^2 + i\gamma q^2\omega) + q^2(c_{\rm s}^2 - i\gamma\omega)$. The first term cancels the denominator of Eq. (3.27), producing the above result for $\boldsymbol{S}_{\rm nonlocal}$. The second term is no longer singular when divided by $q^2$ and is properly viewed as a contribution to the energy flux carried by sound. In the resulting expression (3.31), if one writes $\phi_v(\omega,q^2) = \phi_v(\omega,0) - \left[\phi_v(\omega,q^2) - \phi_v(\omega,0)\right]$ then, strictly speaking, it is only the first $\phi_v(\omega,0)$ term which gives a non-local contribution. But including the additional $\left[\phi_v(\omega,q^2) - \phi_v(\omega,0)\right]$ term in the definition of $S_{\rm nonlocal}$ leads to simpler expressions for the effective sound and diffusion sources discussed below.

[7] Since $\nabla\cdot\boldsymbol{S}_{\rm nonlocal}$ is localized at the quark, the fluxes $\boldsymbol{S}_{\rm sound}$ and $\boldsymbol{S}_{\rm diffusion}$ do provide a decomposition of the energy flux into pieces which are, respectively, longitudinal and transverse everywhere in space excluding the position of the quark, up to terms which fall faster than any inverse power of distance from the

At leading order in the gradient expansion of $\boldsymbol{J}$, the sound and diffusion sources are given by

$$\boldsymbol{J}_{\rm sound} = c_{\rm s}^2 \left(1+\frac{1}{v^2}\right) \nabla F^0 \,, \tag{3.39a}$$

$$\boldsymbol{J}_{\rm diffusion} = \boldsymbol{F} \,, \tag{3.39b}$$

while the leading behavior of the source (3.24) for the energy density is

$$\rho = \nabla \cdot \boldsymbol{F} - \partial_0 F^0 \,. \tag{3.40}$$

Using these sources and the corresponding wave and diffusion equations (3.29), (3.35), and (3.37), the long range behavior of the sound and diffusion modes is completely specified. We note that in the large $N_{\rm c}$ limit the magnitude of the drag force appears as an overall normalization of the perturbation in the stress-energy tensor.

## IV. GRAVITATIONAL DESCRIPTION

According to gauge/string duality, $\mathcal{N}=4$ SYM (on $\mathbb{R}^4$) is equivalent to type IIB string theory on $AdS_5 \times S^5$ [18, 19]. Turning on a non-zero temperature corresponds, in the dual description, to introducing a black brane (a black hole with a flat horizon) into the $AdS_5$ space, leading to the AdS-Schwarzschild geometry described by the metric (2.1).[8]

The addition of massive fundamental representation fields (an $\mathcal{N}=2$ hypermultiplet) to the $\mathcal{N}=4$ SYM theory corresponds, in the dual gravitational description, to the addition of a D7 brane wrapping an $S^3$ of the internal $S^5$ and covering the five dimensional asymptotically AdS space from the boundary down to a minimal radial position $u_m$, which is inversely related to the hypermultiplet mass $M$ for large mass [4].

A single quark moving through the $\mathcal{N}=4$ SYM plasma corresponds, under gauge/string duality, to an open string which runs from the D7 brane down to the black hole horizon [4, 37]. The presence of the string perturbs the geometry via Einstein's equations and the behavior of the metric perturbation near the AdS boundary encodes the change in the SYM stress-energy tensor.[9]

In the $N_{\rm c} \to \infty$ limit, the $5d$ gravitational constant becomes parametrically small and consequently the presence of the string acts as a small perturbation on the AdS-BH geometry. To obtain leading order results in $N_{\rm f}/N_{\rm c} \ll 1$, we write the full metric as $G_{MN} = G^{(0)}_{MN} + h_{MN}$, where $G^{(0)}_{MN}$ is the metric of the AdS-BH geometry given in Eq. (2.1), and then linearize the resulting Einstein equations in the perturbation $h_{MN}$.

According to the AdS/CFT correspondence, the on-shell gravitational action $S_{\rm G}$ is the generating functional for the boundary stress-energy tensor [38, 39, 40]. The metric $G_{MN}$ induces a metric $g_{\mu\nu}$ on the boundary of the AdS-BH geometry [40]. The boundary metric is related to the boundary value of $G_{\mu\nu}$ by a scaling function of the AdS radial coordinate. More specifically, because the metric $G_{MN}$ has a second order pole at the boundary, the boundary metric, which must be regular at the boundary, may be defined as

$$g_{\mu\nu} \equiv \frac{u^2}{L^2}\, G_{\mu\nu}\Big|_{u=\epsilon} \,, \tag{4.1}$$

where the infinitesimal $\epsilon$ will be sent to zero after the required boundary terms needed to properly define the gravitational action are added. The expectation value of the boundary stress tensor is then given by [39, 40]

$$T^{\mu\nu}(x) = \lim_{\epsilon\to 0} \frac{2}{\sqrt{-g(x,\epsilon)}} \, \frac{\delta S_{\rm G}}{\delta g_{\mu\nu}(x,\epsilon)} \,, \tag{4.2}$$

with $g$ denoting the determinant of $g_{\mu\nu}$. Defining for later convenience

$$H_{\mu\nu} \equiv \frac{u^2}{L^2}\, h_{\mu\nu} \,, \tag{4.3}$$

the boundary stress-energy tensor may also be expressed as

$$T^{\mu\nu}(x) = 2 \lim_{\epsilon\to 0} \frac{\delta S_{\rm G}}{\delta H_{\mu\nu}(x,\epsilon)} \,. \tag{4.4}$$

(Since the boundary is flat Minkowski space, the factor of $\sqrt{-g}$ in Eq. (4.2) approaches one as $\epsilon \to 0$ and may be ignored.)

The action for the entire gravitational system is given by

$$S_{\rm G} = S_{\rm EH} + S_{\rm GH} + S_{\rm DBI} + S_{\rm NG} + S_{\rm CT} \,. \tag{4.5}$$

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quark. Because hydrodynamics is only valid sufficiently far from the quark, one may regard $\boldsymbol{S}_{\rm sound}$ and $\boldsymbol{S}_{\rm diffusion}$ as only being defined in $\mathbb{R}^3 \backslash \mathcal{B}$, where $\mathcal{B}$ is a ball surrounding the quark. In such a non-contractible region, the decomposition of a vector field into transverse and longitudinal components is not unique. The above decomposition is a choice for which the resulting effective sources $\boldsymbol{J}_{\rm sound}$ and $\boldsymbol{J}_{\rm diffusion}$ are localized at the quark (*i.e.*, have Fourier transforms analytic in $\boldsymbol{q}$). With that additional condition, this decomposition is effectively unique — alternative choices [such as the replacement of $\phi_v(\omega, q^2)$ by $\phi_v(\omega, 0)$ in the definition (3.31) of $\boldsymbol{S}_{\rm nonlocal}$] merely correspond to adding to $\mathcal{S}_{\rm sound}$ (and subtracting from $\mathcal{S}_{\rm diffusion}$) a localized contribution which falls exponentially with distance from the quark.

[8] Turning on a temperature does not deform the internal $S^5$. Only the AdS-Schwarzschild spacetime will be relevant for the our purposes; the five-sphere will play no role and may be ignored.

[9] The D7 brane also perturbs the geometry. This is a small correction of order $1/N_{\rm c}$ provided the number of flavors $N_{\rm f}$ (which is the same as the number of D7 branes) is held fixed as $N_{\rm c} \to \infty$. This correction encodes the $\mathcal{O}(N_{\rm c})$ contribution of fundamental representation fields to the equilibrium pressure and energy density of the SYM plasma. We ignore this correction (which is subleading relative to the $\mathcal{O}(N_{\rm c}^2)$ contribution of adjoint representation fields) as our goal is the non-equilibrium contribution due to a moving quark.

The first term is the Einstein-Hilbert action,

$$S_{\rm EH} \equiv \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-G}\,(R+2\Lambda)\,, \tag{4.6}$$

where $\kappa_5^2 \equiv 4\pi^2 L^3/N_{\rm c}^2$ is the $5d$ gravitational constant, $G$ is the determinant of $G_{\mu\nu}$ and $\Lambda \equiv 6/L^2$ is the cosmological constant. The next term is the Gibbons-Hawking action,

$$S_{\rm GH} \equiv \frac{1}{2\kappa_5^2} \int d^4x \sqrt{-\gamma}\; 2K\,, \tag{4.7}$$

with $\gamma_{\mu\nu}$ the induced metric on the slice $u = \epsilon$ (and $\gamma$ its determinant), and $K$ the trace of the extrinsic curvature of the slice. This term is needed to obtain an action which makes the Dirichlet problem well defined (*i.e.*, an action depending only on first derivatives of the metric) [41]. The remaining terms are the Dirac-Born-Infeld action $S_{\rm DBI}$ for the D7 brane, the Nambu-Goto action $S_{\rm NG}$ for the string, and a counter-term action $S_{\rm CT}$ defined on the boundary which is required for holographic renormalization [40]. This term cancels poles in $1/\epsilon$ which are generated by the bare gravitational action; the form of the counter-terms is constrained by the requirement of diffeomorphism invariance of the gravitational theory. In Eq. (4.5) all boundary integrals are evaluated on the $u = \epsilon$ slice, with $\epsilon \to 0$ after all terms are combined.

### A. Heavy Quark Effective Theory

In the limit that the quark mass becomes arbitrarily large, its presence in the plasma merely serves as a external source for the SYM fields. This can be made explicit by constructing a heavy quark effective theory (HQET) from the field theory Lagrangian (as given explicitly in Ref. [42]). As we now outline, this procedure has a natural counterpart in the gravitational dual.

The interaction between the string and the D7 brane is governed by the DBI action and the Nambu-Goto action. Fig. 1 shows a cartoon of the string plus D-brane system. As the string moves, a flux of energy and momentum flows down the string to the black hole horizon [3, 4]. This flux of energy and momentum is responsible for the drag force on the quark in the dual boundary theory and is supplied by a $U(1)$ gauge field living on the D7 brane — the strength of which is tuned to match the drag force acting on the quark and thereby maintain a constant velocity. The presence of the string pulls on the D7 brane, deforming it. However, in the large mass limit the trailing string will only deform the D7 brane over length scales of order $1/M$. As $M \to \infty$, the string endpoint approaches the boundary and the size of the region in which the D7 brane is significantly deformed shrinks to zero.[10]

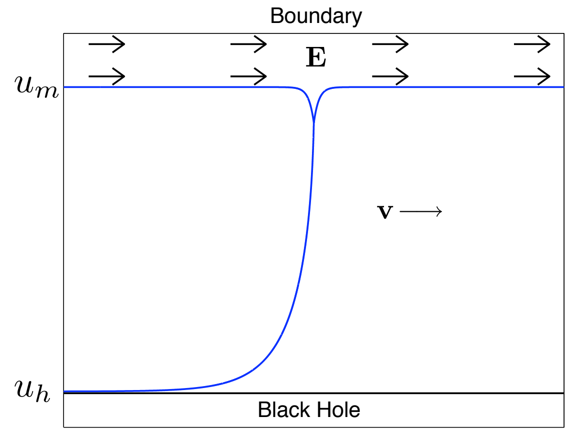


FIG. 1: A cartoon of the string plus D7 brane system in the large mass limit. The D7 brane covers the asymptotically AdS space down to a minimal radial position, away from the string, denoted $u_m$. The trailing string is moving to the right at constant velocity $\boldsymbol{v}$. An energy flux flows down the string toward the black hole horizon, which is located at radial coordinate $u_h$. This energy is supplied by a constant $U(1)$ electric field living on the D7 brane. The D7 brane is deformed in the neighborhood of the endpoint of the string over a length scale of order $1/M$.

Instead of solving the string plus D7 brane system in its entirety, just as in HQET one may consider the large mass limit and construct an effective gravitational theory for the perturbations in the SYM stress-energy tensor. In the effective theory we send the quark mass $M \to \infty$ before performing holographic renormalization of the gravitational action. In doing so, we take $u_m \to 0$ so the D-brane no longer extends inward from the boundary. Furthermore, the deformation of the D-brane due to the string shrinks to a vanishingly small region about the string endpoint (which approaches the boundary).

This suggests that the dynamics of the D7 brane should be irrelevant in the heavy quark limit. This is not quite true, however. Because general relativity is a gauge theory, the construction of a heavy quark effective theory is constrained by the requirement that the effective theory be invariant under infinitesimal diffeomorphisms

$$X_M \to X_M + \xi_M(X)\,, \tag{4.8}$$

where $\xi_M$ is an arbitrary infinitesimal vector field. Because an energy-momentum flux flows from the boundary

[10] This is a bit oversimplified. For a finite mass $M$, the back-reaction of the string on the D7 brane is expected to cause the D-brane embedding to develop a thin narrow tube which reaches all the way down to the horizon [43, 44]. In other words, the fundamental string will be "puffed up" into a tiny tube. However, the resulting dynamics of a sufficiently thin tube is indistinguishable from that of a fundamental string. In the $M \to \infty$ limit, the width of the tube (at any non-zero value of $u$) will shrink to zero and this issue may be ignored.

down the trailing string, gauge invariance at the boundary requires that the $U(1)$ electric field on the D7 brane not be neglected. We therefore construct the effective theory for the perturbations in the SYM stress-energy tensor with the brane effective action

$$S_{\rm DBI}^{\rm eff} = S_{\rm EM}\,, \tag{4.9}$$

where $S_{\rm EM}$ is the Maxwell action for the $U(1)$ electromagnetic field which resides on the boundary.

To complete the definition of the effective theory, we must specify the counter-term action. The counter-terms must cancel $1/\epsilon$ poles in the gravitational action and must be diffeomorphism invariant. The presence of the string ending on the boundary induces $1/\epsilon$ poles in the gravitational action which have point support at the location of the quark. The remaining $1/\epsilon$ poles in the gravitational action are simply those obtained in pure gravity. The appropriate counter-term needed to offset the pure gravity divergences is well known [5, 40, 45] and given by

$$S_{\rm CT} = -\frac{1}{2\kappa_5^2}\,\frac{6}{L}\int d^4x\,\sqrt{\gamma}\,. \tag{4.10}$$

This term is simply proportional to the area of the boundary.

Point-like $1/\epsilon$ divergences with support at the location of the quark are fundamentally different from the other UV divergences — they are physical as the boundary stress-energy tensor *should* contain a $1/\epsilon$ divergence at the location of the quark. This simply reflects the contribution to the stress-energy tensor of an infinitely massive quark,

$$T^{\mu\nu}_{\rm quark} \equiv M\,U^\mu U^\nu \sqrt{1-v^2}\;\delta^3(\boldsymbol{x}{-}\boldsymbol{v}t)\,, \tag{4.11}$$

where $U^\mu$ is the quark's four velocity. The (divergent) coefficient $M$ can either be determined by a matching to the divergent point-like terms in the boundary action or, alternatively, by integrating the energy density of the trailing string and isolating the divergent contribution to find its bare mass. Taking this approach, we find [4]

$$M = \frac{\sqrt{\lambda}}{2\pi\epsilon}\,. \tag{4.12}$$

Instead of adding local counter-terms to the gravitational action to offset the point-like divergence, we simply define the boundary stress-energy tensor via

$$T^{\mu\nu}(x) = \lim_{\epsilon\to 0}\left[2\frac{\delta S_{\rm G}}{\delta H_{\mu\nu}(x,\epsilon)} - T^{\mu\nu}_{\rm quark}(x)\right], \tag{4.13}$$

with $M$ and $\epsilon$ related by (4.12). The resulting stress-energy tensor is finite as $\epsilon \to 0$ and everywhere traceless, including at the position of the quark.[11]

[11] So the Fourier transform $T^{\mu\nu}(\omega,\boldsymbol{q})$ of the stress-energy tensor is traceless for all momenta.

### B. Gauge Invariants

The continuity equation (3.1) and vanishing trace conditions satisfied by the $\mathcal{N}=4$ SYM stress-energy tensor imply, as discussed in section II, that the stress-energy tensor contains five independent degrees of freedom which are conveniently isolated by performing a spacetime Fourier transform and decomposing the stress-energy tensor in terms of helicity variables $\mathcal{T}_0$, $\vec{\mathcal{T}}_1$ and $\overleftrightarrow{\mathcal{T}}_2$.

The five independent degrees of freedom contained in $T_{\mu\nu}$ may be contrasted with the 15 degrees of freedom contained in the metric perturbation $h_{MN}$. Not all of these degrees of freedom are physical, however. The linearized gravitational field equations are invariant under the infinitesimal diffeomorphisms (4.8). Under such transformations, the metric perturbation transforms as

$$h_{MN} \to h_{MN} - D_M\,\xi_N - D_N\,\xi_M\,, \tag{4.14}$$

where $D_M$ is the covariant derivative with respect to the background metric $G^{(0)}_{MN}$. Physical degrees of freedom carried by $h_{MN}$ must be invariant under the above gauge transformations. This limits the number of *independent* physical degrees of freedom carried to the boundary by the metric perturbation $h_{MN}$ to five, matching that of the SYM stress tensor [29].

The correspondences between the number of independent gauge invariant degrees of freedom contained in the metric perturbation $h_{MN}$ and the number of independent degrees of freedom in the SYM stress tensor suggests that the bulk to boundary problem can be formulated directly in terms of gauge invariant degrees of freedom. Using this approach has several advantages [46]. As we show below, the gauge invariants can be chosen to satisfy decoupled equations of motion. Furthermore, as we show in Section IV D, the on-shell gravitational action can be expressed in terms of gauge invariant degrees of freedom plus computable boundary terms (which are due to the non-conservation of the SYM stress tensor at the location of the quark).

Gauge invariants can be constructed out of the Fourier mode amplitudes $h_{MN}(u;\omega,\boldsymbol{q})$ and classified according to their helicity under rotations about the $\hat{q}$ axis [46]. As we shall discuss in detail below, the determination of the SYM stress-energy tensor requires the construction of a scalar gauge invariant $Z_0$, a vector gauge invariant $\vec{Z}_1$, and a tensor gauge invariant $\overleftrightarrow{Z}_2$. The choice of gauge invariants $Z_s$ is not unique. However, all gauge invariants of a common helicity carry the same information to the boundary.[12] We are therefore free to chose any convenient set of gauge invariants.

[12] This may be verified by analyzing the asymptotic behavior of both the linearized Einstein field equations and the various gauge invariants of the same helicity near the boundary. Doing so, one finds that the boundary values of gauge invariants (with the same helicity) are linearly related.

Let $\mathcal{H}_{MN}$ denote the components of $H_{MN}$ in the polarization frame. Using Eq. (4.14) it is easy to see that the following five degrees of freedom are gauge invariant

$$\begin{aligned} Z_0 &\equiv q^2\,\mathcal{H}_{00} + 2\omega q\,\mathcal{H}_{0q} + \omega^2\,\mathcal{H}_{qq}\,, \\ &\quad + \tfrac{1}{2}\left[(2-f)\,q^2-\omega^2\right]\mathcal{H}_{aa}\,, && (4.15a)\\ \vec{Z}_1 &\equiv (\mathcal{H}'_{0a} - i\omega\,\mathcal{H}_{a5})\,\hat{\epsilon}_a\,, && (4.15b)\\ \overleftrightarrow{Z}_2 &\equiv \left(\mathcal{H}_{ab} - \tfrac{1}{2}\mathcal{H}_{cc}\,\delta_{ab}\right)\hat{\epsilon}_a\otimes\hat{\epsilon}_b\,, && (4.15c)\end{aligned}$$

where sums over repeated indices are implied and primes denotes differentiation with respect to $u$.[13]

### C. Equations of Motion

The equations of motion for the invariants $Z_s$ follow from the linearized Einstein field equations. The full Einstein field equations are

$$R_{MN} - \tfrac{1}{2}G_{MN}\,(R+2\Lambda) = \kappa_5^2\,t_{MN}\,, \tag{4.16}$$

where $t_{MN}$ is the $5d$ stress-energy tensor of the trailing string. Writing

$$G_{MN} = G^{(0)}_{MN} + h_{MN}\,, \tag{4.17}$$

where $G^{(0)}_{MN}$ is the AdS-BH metric, and expanding the left hand side of the field equations (4.16) in the perturbation $h_{MN}$, produces the linearized equations,

$$\begin{aligned} &- D^2\,h_{MN} + 2D^P D_{(M}h_{N)P} - D_M D_N\,h + \tfrac{8}{L^2}\,h_{MN} \\ &+ \left(D^2 h - D^P D^Q\,h_{PQ} - \tfrac{4}{L^2}\,h\right)G^{(0)}_{MN} = 2\kappa_5^2\,t_{MN}\,, \end{aligned} \tag{4.18}$$

where $h \equiv h^M_{\;M}$.

The trailing string profile is determined by minimizing the Nambu-Goto action for a stationary, constant velocity profile. The result is [3, 4]

$$\boldsymbol{x}(t,u) = \boldsymbol{v}\,t + \boldsymbol{x}_{\rm string}(u)\,, \tag{4.19}$$

with

$$\boldsymbol{x}_{\rm string}(u) \equiv \frac{\boldsymbol{v}\,u_h}{2}\left[\tan^{-1}\!\left(\frac{u}{u_h}\right) + \tfrac{1}{2}\ln\left(\frac{u_h-u}{u_h+u}\right)\right]. \tag{4.20}$$

The $5d$ stress-energy tensor for the trailing string is [26]

$$\begin{aligned} t_{00} &= s\left(f + v^2u^4u_h^{-4}\right), & t_{55} &= s\,(f+v^2)\,f^{-2}\,, && (4.21a)\\ t_{0i} &= -s\,v_i\,, & t_{ij} &= s\,v_iv_j\,, && (4.21b)\\ t_{05} &= -s\,v^2\,f^{-1}u^2u_h^{-2}\,, & t_{i5} &= s\,v_i\,f^{-1}u^2u_h^{-2}\,, && (4.21c)\end{aligned}$$

where

$$s(u) \equiv \frac{u\,\sqrt{\lambda}}{2\pi L^3\sqrt{1-v^2}}\,\delta^3(\boldsymbol{x} - \boldsymbol{v}t - \boldsymbol{x}_{\rm string}(u))\,. \tag{4.22}$$

We denote the components of $t_{\mu\nu}$ in the polarization frame by $\boldsymbol{t}_{\mu\nu}$.

Because the background geometry is invariant under spatial rotations, the differential operators on the LHS of Eq. (4.18) cannot couple linear combinations of $\mathcal{H}_{MN}$ which transform with different helicities under rotations about the $\hat{\boldsymbol{q}}$ axis. Consequently, the field equations (4.18) reduce to three coupled sets of ordinary differential equations labeled by their helicity. Although it is not necessary, working in the gauge in which $h_{5N} = 0$ for all $N$ simplifies the extraction of decoupled equations for the gauge invariants $Z_s$. The scalar set of equations then reduces to four second order and three first order equations, while the vector set of equations reduces to two second order and one first order equation. However, not all of these equations are independent — some of the second order equations can be derived from the first order equations. One may use this observation to reduce the number of required second order equations of motion.

#### 1. *Tensor Mode*

The components of $\mathcal{H}_{\mu\nu}$ which transform with helicity two under rotations about the $\hat{q}$ axis are the traceless part of $\mathcal{H}_{ab}$. The $ab$ components of the linearized equations (4.18) may be written explicitly as

$$\begin{aligned} &-f\,\mathcal{K}''_{ab} - \frac{uf'-3f}{u}\,\mathcal{K}'_{ab} + \frac{q^2f-\omega^2}{f}\,\mathcal{K}_{ab} \\ &\quad - \left[f'\left(\mathcal{H}_{00}/2f\right)' - \widetilde{Q}^\mu\widetilde{Q}^\nu\,\mathcal{H}_{\mu\nu}\right]\delta_{ab} = 2\kappa_5^2\,\boldsymbol{t}_{ab}\,, \end{aligned} \tag{4.23}$$

where $\widetilde{Q}^\mu \equiv (\omega/f, \boldsymbol{q})$ and

$$\mathcal{K}_{ab} \equiv \mathcal{H}_{ab} + (\mathcal{H}_{00}/f - \mathcal{H}_{ii})\,\delta_{ab}\,. \tag{4.24}$$

The equation of motion for the helicity 2 gauge invariant $\overleftrightarrow{Z}_2$ immediately follows from the traceless part of Eq. (4.23),

$$\overleftrightarrow{Z}''_2 + A_2\,\overleftrightarrow{Z}'_2 + B_2\,\overleftrightarrow{Z}_2 = \overleftrightarrow{S}_2\,, \tag{4.25}$$

where

$$\begin{aligned} A_2 &\equiv \frac{uf'-3f}{uf}\,, && (4.26a)\\ B_2 &\equiv -\frac{q^2f-\omega^2}{f^2}\,, && (4.26b)\\ \overleftrightarrow{S}_2 &\equiv -\frac{2\kappa_5^2}{f}\,(\boldsymbol{t}_{ab} - \tfrac{1}{2}\,\boldsymbol{t}_{cc}\,\delta_{ab})\,\hat{\epsilon}_a\otimes\hat{\epsilon}_b\,. && (4.26c)\end{aligned}$$

[13] Using the differential helicity one invariant (4.15b) instead of an equivalent non-differential invariant turns out to be more convenient for technical reasons involving the numerical stability of the resulting ordinary differential equation.

Inserting the string stress-energy tensor (4.21) into expression (4.26c) for the source $\overleftrightarrow{S}_2$ gives

$$\overleftrightarrow{S}_2 = -\frac{\kappa_5^2\sqrt{\lambda}}{2\pi L^3}\,\frac{uv^2q_\perp^2}{q^2f\sqrt{1-v^2}}\,(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q}) \times e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_{\rm string}}\,(\hat\epsilon_1\otimes\hat\epsilon_1-\hat\epsilon_2\otimes\hat\epsilon_2)\,. \tag{4.27}$$

### 2. Vector Mode

The components of $\mathcal{H}_{\mu\nu}$ which transform with helicity one under rotations about the $\hat q$ axis are $\mathcal{H}_{0a}$, $\mathcal{H}_{aq}$ and $\mathcal{H}_{a5}$ (which has been set to zero). The equations of motion for these quantities contain two second order equations and one first order equation. For the derivation of the equation of motion for $\vec Z_1$, it is sufficient to use one second order equation (which is independent of the first order equation) and the first order equation. We use the following components of the linearized field equations (4.18)

$$-f\,\mathcal{H}_{0a}'' + \frac{3f}{u}\,\mathcal{H}_{0a}' + q^2\,\mathcal{H}_{0a} + q\omega\,\mathcal{H}_{aq} = 2\kappa_5^2\,\boldsymbol{t}_{0a}, \tag{4.28}$$

$$\frac{1}{f}\left(i\omega\,\mathcal{H}_{0a}' + iqf\,\mathcal{H}_{aq}'\right) = 2\kappa_5^2\,\boldsymbol{t}_{a5}\,. \tag{4.29}$$

Using these equations, we find the equation of motion for the helicity one invariant $\vec Z_1$

$$\vec Z_1'' + A_1\,\vec Z_1' + B_1\,\vec Z_1 = \vec S_1\,, \tag{4.30}$$

where

$$A_1 \equiv \frac{uf'-3f}{uf}\,, \tag{4.31a}$$

$$B_1 \equiv \frac{3f^2 - u\left(uq^2+3f'\right)f + u^2\omega^2}{u^2f^2}\,, \tag{4.31b}$$

$$\vec S_1 \equiv \frac{2\kappa_5^2}{f}\left[\boldsymbol{t}_{0a}' + i\omega\,\boldsymbol{t}_{a5}\right]\hat\epsilon_a\,. \tag{4.31c}$$

Substituting the string stress-energy tensor (4.21) into Eq. (4.31c), gives

$$\vec S_1 = \frac{\kappa_5^2\sqrt{\lambda}}{\pi L^3}\,\frac{vq_\perp}{qf\sqrt{1-v^2}}\,(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\,e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_{\rm string}}\,\hat\epsilon_1\,. \tag{4.32}$$

### 3. Scalar Mode

The (non-vanishing) components of $\mathcal{H}_{\mu\nu}$ which transform like scalars under rotations about the $\hat q$ axis are $\mathcal{H}_{00}$, $\mathcal{H}_{0q}$, $\mathcal{H}_{qq}$ and $\mathcal{H}_{aa}$. The equations of motion for these quantities contain four second order equations and three first order equations. For the derivation of the equation of motion for $Z_0$, it is sufficient to use one second order equation (which is independent of the three first order equations) and the three first order equations. By taking suitable linear combinations of the components of the linearized field equations (4.18), we find the following equations of motion:

$$-\mathcal{H}_{00}'' + \frac{6f+uf'}{2uf}\,\mathcal{H}_{00}' + \frac{f'}{3}\,\mathcal{H}_{ii}' + \frac{4q^2f-3f'^2}{6f^2}\,\mathcal{H}_{00} + \frac{q^2f+2\omega^2}{3f}\,\mathcal{H}_{ii} - \frac{q^2}{3}\,\mathcal{H}_{qq} + \frac{4q\omega}{3f}\,\mathcal{H}_{0q} = \frac{2\kappa_5^2}{3f}\left(2\,\boldsymbol{t}_{00}+f\,\boldsymbol{t}_{ii}\right), \tag{4.33a}$$

$$3\mathcal{H}_{00}' + \tfrac{1}{2}(uf'-6f)\,\mathcal{H}_{ii}' + \frac{u\omega^2}{f}\,\mathcal{H}_{ii} + uq^2\,\mathcal{H}_{aa} + \frac{q^2u-3f'}{f}\,\mathcal{H}_{00} + \frac{2u\omega q}{f}\,\mathcal{H}_{0q} = 2uf\kappa_5^2\,\boldsymbol{t}_{55}\,, \tag{4.33b}$$

$$i\omega\,\mathcal{H}_{0q}' + iq\,\mathcal{H}_{00}' - iqf\,\mathcal{H}_{aa}' - \frac{iqf'}{2f}\,\mathcal{H}_{00} = 2f\kappa_5^2\,\boldsymbol{t}_{q5}\,, \tag{4.33c}$$

$$i\omega\,\mathcal{H}_{ii}' + iq\,\mathcal{H}_{0q}' - \frac{i\omega f'}{2f}\,\mathcal{H}_{ii} - \frac{iqf'}{f}\,H_{0q} = 2\kappa_5^2\,\boldsymbol{t}_{05}\,. \tag{4.33d}$$

It is straightforward (but tedious) to derive the equation of motion of $Z_0$ from the above equations. The result has the form

$$Z_0'' + A_0\,Z_0' + B_0\,Z_1 = S_0\,. \tag{4.34}$$

The coefficients $A_0$ and $B_0$ and the source $S_0$ can be determined by substituting the definition of $Z_0$ into the above ansatz and then using the above equations of motion (4.33a)–(4.33d) to eliminate all derivatives except $\mathcal{H}_{00}'$. Demanding that the coefficient $\mathcal{H}_{00}'$ vanish then determines $A_0$. Once $A_0$ is determined, it is straightforward to read off the coefficient $B_0$ and the source $S_0$. One finds

$$A_0 = \frac{1}{u}\left[1 + \frac{uf'}{f} + \frac{24\left(q^2f-\omega^2\right)}{q^2\left(uf'-6f\right)+6\omega^2}\right], \tag{4.35a}$$

$$B_0 = \frac{1}{f}\left[-q^2 + \frac{\omega^2}{f} - \frac{32\,q^2u^6u_h^{-8}}{q^2(uf'-6f)+6\omega^2}\right]. \tag{4.35b}$$

The expression for the source $S_0$ in terms of a general string stress-energy tensor $\boldsymbol{t}_{MN}$ is sufficiently lengthy that we will not give it here. But substituting the explicit form of the string stress-energy tensor (4.21) produces a relatively compact result,

$$S_0 \equiv \frac{\kappa_5^2\sqrt{\lambda}}{6\pi L^3}\,\frac{q^2\left(v^2+2\right)-3\omega^2}{q^2\sqrt{1-v^2}} \times \frac{u[q^4u^8 + 48iq^2\omega u_h^2u^5 - 9(q^2-\omega^2)^2u_h^8]}{f\left(fq^2+2q^2-3\omega^2\right)u_h^8} \times (2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\;e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_{\rm string}}\,. \tag{4.36}$$

### D. SYM Stress Tensor

The variation of the on-shell gravitational action may be expressed as

$$\delta S_{\rm G} = \delta S_{\rm horizon} + \delta S_{\rm B}\,, \tag{4.37}$$

where $S_{\rm B}$ is a surface term at the boundary, and $S_{\rm horizon}$ is a surface term at the horizon. Following Ref. [47], we neglect $S_{\rm horizon}$.[14] We show in Appendix A that

$$\begin{aligned}\delta S_{\rm B} = \int_{u=\epsilon} \frac{d^4q}{(2\pi)^4} \Big[\, & \mathcal{A}_0\, \delta Z_0^\dagger\, \partial_u\, Z_0 + \mathcal{A}_1\, \delta \vec{\mathcal{Z}}_1^\dagger\, \vec{\mathcal{Z}}_1 \\ & + \mathcal{A}_2\, \mathrm{tr}\, \delta \overleftrightarrow{Z}_2^\dagger\, \partial_u\, \overleftrightarrow{Z}_2 + \tfrac{1}{2}\, \delta\mathcal{H}_{\mu\nu}^\dagger\, \mathcal{T}_{\rm eq}^{\mu\nu} \\ & + \tfrac{1}{2}\, \delta\mathcal{H}_{\mu\nu}^\dagger\, \mathcal{J}^{\mu\nu} \Big] + \delta S_{\rm EM} + \mathcal{O}(\epsilon)\,,\end{aligned} \tag{4.38}$$

with

$$\mathcal{A}_0 = \frac{L^3}{6\kappa_5^2 (q^2{-}\omega^2)^2}\, \frac{1}{u^3}\,, \tag{4.39a}$$

$$\mathcal{A}_1 = -\frac{L^3}{2\kappa_5^2\, q}\, \frac{1}{u^3}\,, \tag{4.39b}$$

$$\mathcal{A}_2 = \frac{L^3}{4\kappa_5^2}\, \frac{1}{u^3}\,, \tag{4.39c}$$

and

$$\vec{\mathcal{Z}}_1 \equiv (q\mathcal{H}_{0a} + \omega\mathcal{H}_{aq})\, \hat{\epsilon}^a\,. \tag{4.40}$$

In the fourth term of (4.38), $T_{\rm eq}^{\mu\nu}$ denotes the stress-energy tensor of the equilibrium $\mathcal{N}=4$ SYM plasma,

$$T_{\rm eq}^{\mu\nu} = \tfrac{3}{8} N_{\rm c}^2 \pi^2 T^4\, \mathrm{diag}(1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})\,, \tag{4.41}$$

with $\mathcal{T}_{\rm eq}^{\mu\nu}$ its components in the polarization frame. In the fifth term of (4.38), the symmetric tensor $\mathcal{J}^{\mu\nu}$ is a linear combination of components of the string stress-energy tensor evaluated at the boundary.[15] Explicitly,

$$\mathcal{J}^{00} = \mathcal{C} \left\{ \frac{q^2}{u} - \frac{i\omega \left[ q^2 \left(5v^2{+}1\right) - 3\omega^2 \left(v^2{+}1\right)\right]}{u_h^2\, (1{-}v^2)\, (q^2{-}\omega^2)} \right\}, \tag{4.42a}$$

$$\mathcal{J}^{qq} = \mathcal{C} \left\{ \frac{\omega^2}{u} - \frac{i\omega \left[ 3q^2 \left(v^2{+}1\right) - \left(v^2{+}5\right) \omega^2\right]}{u_h^2\, (1{-}v^2)\, (q^2{-}\omega^2)} \right\}, \tag{4.42b}$$

$$\mathcal{J}^{0q} = \mathcal{C} \left\{ \frac{q\omega}{u} - \frac{i \left[ 3v^2 q^4 + \left(1{-}v^2\right) \omega^2 q^2 - 3\omega^4\right]}{q u_h^2\, (1{-}v^2)\, (q^2{-}\omega^2)} \right\}, \tag{4.42c}$$

$$\mathcal{J}^{aa} = \mathcal{C} \left[ -\frac{q^2{-}\omega^2}{u} + \frac{i\omega}{u_h^2} \right], \tag{4.42d}$$

$$\mathcal{J}^{aq} = \mathcal{C} \left[ -\frac{3i\, v\, q_\perp \left(q^2{-}\omega^2\right)}{q^2 u_h^2\, (1{-}v^2)} \right], \tag{4.42e}$$

with

$$\mathcal{C} \equiv \frac{\sqrt{\lambda}}{6\pi}\, \frac{\sqrt{1{-}v^2}}{q^2{-}\omega^2}\, (2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q}). \tag{4.43}$$

All other components of $\mathcal{J}^{\mu\nu}$ vanish. In Eq. (4.42d), no sum on $a$ is implied.

On the boundary, the action of diffeomorphisms on the metric perturbation takes the simple form

$$H_{\mu\nu} \to H_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\,. \tag{4.44}$$

The first four terms in the boundary action (4.38) are manifestly invariant. The fifth term however violates diffeomorphism invariance. One may easily verify that $\mathcal{J}^{\mu\nu}$ satisfies

$$iq_\mu\, \mathcal{J}^{\mu\nu} = \mathcal{F}^\nu\,, \tag{4.45}$$

where

$$\mathcal{F}^0 = \frac{v^2\sqrt{\lambda}}{2\pi u_h^2 \sqrt{1{-}v^2}}\, (2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,, \tag{4.46a}$$

$$\mathcal{F}^q = \frac{(\hat{q}\cdot\boldsymbol{v})\sqrt{\lambda}}{2\pi u_h^2 \sqrt{1{-}v^2}}\, (2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,, \tag{4.46b}$$

$$\mathcal{F}^a = \frac{(\hat{\epsilon}_a\cdot\boldsymbol{v})\sqrt{\lambda}}{2\pi u_h^2 \sqrt{1{-}v^2}}\, (2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,. \tag{4.46c}$$

These are precisely the components of the external force density $F^\nu$ in the polarization basis. The violation of diffeomorphism (or gauge) invariance only occurs at the location of the string endpoint. The presence of $\mathcal{J}^{\mu\nu}$ in the boundary action reflects the flux of energy-momentum from the boundary into the string. This flux comes from the $U(1)$ electric field living on the D7 brane. By definition,

$$\delta S_{\rm EM} = \int \frac{d^4q}{(2\pi)^4}\, \tfrac{1}{2}\, \delta\mathcal{H}_{\mu\nu}\, \mathcal{T}_{\rm EM}^{\mu\nu}\,, \tag{4.47}$$

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[14] This can be understood as follows. As we discuss below, causality implies that near the horizon the gauge invariants behave as $Z_s \sim (u{-}u_h)^{-i\omega u_h/4}$. To make the $u \to u_h$ limit of this quantity (and $S_{\rm horizon}$) meaningful, the frequency can be infinitesimally analytically continued into the complex plane. Causality dictates the required direction of continuation. For retarded boundary conditions, one must send $\omega$ into the upper half plane, $\omega \to \omega{+}i\delta$. With such an infinitesimal continuation understood, the gauge invariants $Z_s$ vanish at the horizon.

[15] See Eqs. (A27) and (A31) for the definitions expressing $\mathcal{J}^{\mu\nu}$ in terms of $t_{MN}$.

where $\mathcal{T}^{\mu\nu}_{\rm EM}$ are the components of the electromagnetic stress-energy tensor in the polarization frame. By construction, the strength of the $U(1)$ electric field is tuned to offset the drag force and maintain a constant velocity of the quark. It therefore follows that we must have

$$iq_\mu \mathcal{T}^{\mu\nu}_{\rm EM} = -\mathcal{F}^\nu. \tag{4.48}$$

Consequently, the total boundary action is gauge (diffeomorphism) invariant.

Given the boundary action (4.38), one can evaluate the boundary stress-energy tensor via the definition (4.13). As discussed earlier, it is convenient to express the SYM stress-energy tensor in terms of its helicity variables $\mathcal{T}_s$. Let $\mathcal{T}_s^{\rm quark}$ and $\mathcal{T}_s^{\rm eq}$ denote the helicity variables of $T^{\mu\nu}_{\rm quark}$ and $T^{\mu\nu}_{\rm eq}$, respectively. Neglecting the contribution from the $U(1)$ electromagnetic field, a short exercise yields

$$\mathcal{T}_0 = \lim_{u\to 0}\Big[2q^2\mathcal{A}_0\,\partial_u Z_0 + \mathcal{J}^{00} + \mathcal{T}_0^{\rm eq} - \mathcal{T}_0^{\rm quark}\Big], \tag{4.49a}$$

$$\vec{\mathcal{T}}_1 = \lim_{u\to 0}\Big[q\mathcal{A}_1\vec{Z}_1 + \vec{\mathcal{T}}_1^{\rm eq} - \vec{\mathcal{T}}_1^{\rm quark}\Big], \tag{4.49b}$$

$$\overleftrightarrow{\mathcal{T}}_2 = \lim_{u\to 0}\Big[2\mathcal{A}_2\,\partial_u \overleftrightarrow{Z}_2 + \overleftrightarrow{\mathcal{T}}_2^{\rm eq} - \overleftrightarrow{\mathcal{T}}_2^{\rm quark}\Big]. \tag{4.49c}$$

We now simplify the above expressions for the helicity variables $\mathcal{T}_s$. The gauge invariants are solutions of their respective differential equations (4.25), (4.30) and (4.34), and vanish at the boundary. This implies that they have the power series expansions,

$$Z_0 = u^3 Z_0^{(3)} + u^4 Z_0^{(4)} + \cdots, \tag{4.50a}$$

$$\vec{Z}_1 = u^2 \vec{Z}_1^{(2)} + u^3 \vec{Z}_1^{(3)} + \cdots, \tag{4.50b}$$

$$\overleftrightarrow{Z}_2 = u^3 \overleftrightarrow{Z}_2^{(3)} + u^4 \overleftrightarrow{Z}_2^{(4)} + \cdots. \tag{4.50c}$$

The leading terms in these expansions are temperature independent. Plugging the expansions into the respective equations one finds

$$Z_0^{(3)} = \frac{\kappa_5^2\sqrt{\lambda}}{6\pi L^3}\,\frac{\left[q^2\left(v^2{+}2\right)-3\omega^2\right]\left(q^2{-}\omega^2\right)}{q^2\sqrt{1{-}v^2}} \times (2\pi)\delta(\omega - \boldsymbol{v}\cdot\boldsymbol{q}), \tag{4.51a}$$

$$\vec{Z}_1^{(2)} = -\frac{\kappa_5^2\sqrt{\lambda}}{\pi L^3}\,\frac{vq_\perp}{q\sqrt{1{-}v^2}}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,\hat{\epsilon}_1, \tag{4.51b}$$

$$\overleftrightarrow{Z}_2^{(3)} = \frac{\kappa_5^2\sqrt{\lambda}}{6\pi L^3}\,\frac{v^2 q_\perp^2}{q^2\sqrt{1-v^2}}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q}) \times (\hat{\epsilon}_1\otimes\hat{\epsilon}_1 - \hat{\epsilon}_2\otimes\hat{\epsilon}_2). \tag{4.51c}$$

From the definition (4.11) of $T^{\mu\nu}_{\rm quark}$ and our choice (2.2) of polarization vectors, it is easy to see that

$$\mathcal{T}_0^{\rm quark} = \frac{M}{\sqrt{1{-}v^2}}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q}), \tag{4.52a}$$

$$\vec{\mathcal{T}}_1^{\rm quark} = \frac{M}{\sqrt{1{-}v^2}}\,\frac{vq_\perp}{q}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q})\,\hat{\epsilon}_1, \tag{4.52b}$$

$$\overleftrightarrow{\mathcal{T}}_2^{\rm quark} = \frac{M}{\sqrt{1{-}v^2}}\,\frac{(vq_\perp)^2}{2q^2}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q}) \times (\hat{\epsilon}_1\otimes\hat{\epsilon}_1 - \hat{\epsilon}_2\otimes\hat{\epsilon}_2), \tag{4.52c}$$

where $M \equiv \sqrt{\lambda}/(2\pi\epsilon)$ is the quark mass which is being sent to infinity. The helicity components of the equilibrium stress-energy tensor are just

$$\mathcal{T}_0^{\rm eq} = \frac{3}{8}\,N_{\rm c}^2\pi^2T^4, \quad \vec{\mathcal{T}}_1^{\rm eq} = 0, \quad \overleftrightarrow{\mathcal{T}}_2^{\rm eq} = 0. \tag{4.53}$$

Putting it all together we have[16]

$$\mathcal{T}_0 = \frac{4q^2L^3}{3\kappa_5^2(q^2{-}\omega^2)^2}\,Z_0^{(4)} + \mathcal{D} + \mathcal{T}_0^{\rm eq}, \tag{4.54a}$$

$$\vec{\mathcal{T}}_1 = -\frac{L^3}{2\kappa_5^2}\,\vec{Z}_1^{(3)}, \tag{4.54b}$$

$$\overleftrightarrow{\mathcal{T}}_2 = \frac{2L^3}{\kappa_5^2}\,\overleftrightarrow{Z}_2^{(4)}. \tag{4.54c}$$

where

$$\begin{aligned}\mathcal{D} &\equiv -\mathcal{C}\,\frac{i\omega\left[q^2\left(5v^2{+}1\right)-3\left(v^2{-}1\right)\omega^2\right]}{u_h^2\,(1{-}v^2)\,(q^2{-}\omega^2)} \\ &= -\frac{\sqrt{\lambda}}{6\pi}\,\sqrt{1{-}v^2}\,(2\pi)\delta(\omega{-}\boldsymbol{v}\cdot\boldsymbol{q}) \\ &\qquad \times \frac{i\omega\left[q^2\left(5v^2{+}1\right)-3\left(v^2{-}1\right)\omega^2\right]}{u_h^2\,(1{-}v^2)\,(q^2{-}\omega^2)^2}.\end{aligned} \tag{4.55}$$

Note that, after subtracting $\mathcal{T}_s^{\rm quark}$, all divergent quantities have canceled.

Below, we will focus on the temperature dependent perturbation in the stress-energy tensor due to the moving quark,[17]

$$\Delta T^{\mu\nu} \equiv T^{\mu\nu} - T^{\mu\nu}_{\rm eq} - T^{\mu\nu}\Big|_{T=0}. \tag{4.56}$$

That is, we subtract the zero temperature stress-energy of the quark as well as the stress-energy of the equilibrium plasma. We also define

$$\Delta\mathcal{E} \equiv \Delta T^{00}, \tag{4.57a}$$

$$\Delta S_i \equiv \Delta T^{0i}, \tag{4.57b}$$

which are the temperature dependent perturbations in the energy density and energy flux, respectively.

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[16] These expressions for the $\mathcal{T}_s$ can also be obtained by solving the complete set of field equations for the metric perturbation $H_{\mu\nu}$. Near the boundary, $H_{\mu\nu} = u^3\,H^{(3)}_{\mu\nu} + u^4\,H^{(4)}_{\mu\nu} + \cdots$. In a gauge in which $h_{5M} = 0$, the SYM stress-energy tensor is given by $T_{\mu\nu} = T^{\rm eq}_{\mu\nu} + \frac{2L^3}{\kappa_5^2}\,H^{(4)}_{\mu\nu}$ [40]. This approach was taken in Refs. [26, 29, 48]. Our treatment above has the virtue of being self-contained and very explicit about the treatment of contributions from the quark and the external electric field.

[17] The zero-temperature stress energy tensor is completely determined, up to the overall normalization, by conformal invariance. In the rest frame of the quark it is given by $T^{\mu\nu}(x)\Big|_{T=0} = \sqrt{\lambda}\,{\rm diag}(1,\frac{1}{3},\frac{1}{3},\frac{1}{3})/(12\pi^2\boldsymbol{x}^4)$ [27].

As discussed in section II, the symmetric tensor $\Delta T^{\mu\nu}$ is determined by the energy-momentum conservation equation (3.1) it obeys together with the perturbations in the associated helicity variables,

$$\Delta\mathcal{T}_0 = \frac{4q^2L^3}{3\kappa_5^2(q^2-\omega^2)^2}\,\Delta Z_0^{(4)} + \mathcal{D}\,, \tag{4.58a}$$

$$\Delta\vec{\mathcal{T}}_1 = -\frac{L^3}{2\kappa_5^2}\,\Delta\vec{Z}_1^{(3)}\,, \tag{4.58b}$$

$$\Delta\overleftrightarrow{\mathcal{T}}_2 = \frac{2L^3}{\kappa_5^2}\,\Delta\overleftrightarrow{Z}_2^{(4)}\,, \tag{4.58c}$$

where $\Delta Z_0^{(4)}$, $\Delta\vec{Z}_1^{(3)}$, and $\Delta\overleftrightarrow{Z}_2^{(4)}$ are the values of $Z_0^{(4)}$, $\vec{Z}_1^{(3)}$ and $\overleftrightarrow{Z}_2^{(4)}$, respectively, at temperature $T$ minus zero temperature.

### E. Numerical Solutions

One must solve the inhomogeneous differential equations for the $Z_s$, Eqs. (4.25), (4.30) and (4.34), with appropriate boundary conditions at the horizon and at the boundary. To do so, we construct Green's functions $G_s(u,u')$ out of homogeneous solutions,

$$G_s(u,u') = g_s^<(u_<)\, g_s^>(u_>)\;/\;W_s(u')\,, \tag{4.59}$$

with $W_s(u)$ the Wronskian of $g_s^<$ and $g_s^>$, and convolve with the source,

$$Z_s(u) = \int_0^{u_h} du'\, G_s(u,u')\, S_s(u')\,. \tag{4.60}$$

The appropriate homogeneous solutions to use are dictated by the boundary conditions. The differential operators in Eqs. (4.25), (4.30) and (4.34) have singular points at $u=0$ and $u=u_h$. The indicial exponents at $u=0$ are non-negative integers and those at $u=u_h$ are $\pm i\omega u_h/4$. Vanishing of the metric perturbation at the boundary requires that $g_s^<(0)=0$, while the requirement that the black hole not radiate [47] implies that $g_s^>(u) \sim (u-u_h)^{-i\omega u_h/4}$ as $u\to u_h$.

#### 1. Tensor Mode

The indicial exponents of the helicity two differential equation (4.25) at $u=0$ are 0 and 4. Hence the homogeneous solution $g_2^<$ must have the asymptotic behavior $g_2^<(u) \sim u^4$ as $u\to 0$. We fix the (arbitrary) overall normalization by requiring that $\lim_{u\to 0} g_2^<(u)/u^4 = 1$.

At zero temperature, the homogeneous solutions to the helicity two equation are

$$g^>_{2,T=0}(u) = u^2\, K_2(u\sqrt{q^2-\omega^2})\,, \tag{4.61a}$$

$$g^<_{2,T=0}(u) = \frac{8u^2}{q^2-\omega^2}\, I_2(u\sqrt{q^2-\omega^2})\,, \tag{4.61b}$$

where $K_2$ and $I_2$ are modified Bessel functions. The corresponding zero temperature Wronskian is

$$W_{2,T=0}(u) = -\frac{8u^3}{q^2-\omega^2}\,. \tag{4.62}$$

The radial coordinate of zero temperature AdS space (using the metric (2.1) with $u_h=\infty$) runs over the interval $(0,\infty)$. To subtract zero temperature contributions to the stress-energy tensor it will be convenient to map this interval onto $(0,u_h)$ with a change of variables

$$\xi(u) = \frac{u}{f(u)}\,. \tag{4.63}$$

Putting everything together, we have

$$\begin{aligned}\Delta\overleftrightarrow{Z}_2^{(4)} = \int_0^{u_h} du\,\Big\{&\frac{g_2^>(u)}{W_2(u)}\,\overleftrightarrow{S}_2(u) \\ &+ J(u)\,\frac{q^2-\omega^2}{8\,\xi(u)}\,K_2\Big(\xi(u)\sqrt{q^2-\omega^2}\Big)\,\overleftrightarrow{S}_{2,T=0}(\xi(u))\Big\}\,,\end{aligned} \tag{4.64}$$

where $J \equiv d\xi/du = (f-uf')/f^2$ is the Jacobian for the change of variables (4.63).

#### 2. Vector Mode

The indicial exponents of the helicity one differential equation (4.30) at $u=0$ are 0 and 3. Vanishing of the metric perturbation at the boundary thus requires that $g_1^<(u)\sim u^3$ as $u\to 0$. The overall normalization of $g_1^<$ is fixed by requiring $\lim_{u\to 0} g_1^<(u)/u^3 \equiv 1$.

At zero temperature, the homogeneous solutions to the helicity one equation are

$$g^>_{1,T=0}(u) = u^2 K_1(u\sqrt{q^2-\omega^2})\,, \tag{4.65a}$$

$$g^<_{1,T=0}(u) = \frac{2u^2}{\sqrt{q^2-\omega^2}}\, I_1(u\sqrt{q^2-\omega^2})\,, \tag{4.65b}$$

and their Wronskian is

$$W_{1,T=0}(u) = -\frac{2u^3}{\sqrt{q^2-\omega^2}}\,. \tag{4.66}$$

We therefore find

$$\begin{aligned}\Delta\vec{Z}_1^{(3)} = \int_0^{u_h} du\,\Big\{&\frac{g_0^>(u)}{W_0(u)}\,\vec{S}_1(u) \\ &+ J(u)\,\frac{\sqrt{q^2-\omega^2}}{2\,\xi(u)}\,K_1\Big(\xi(u)\sqrt{q^2-\omega^2}\Big)\,\vec{S}_{1,T=0}(\xi(u))\Big\}.\end{aligned} \tag{4.67}$$

#### 3. Scalar Mode

Finally, the exponents of the helicity zero differential equation (4.34) at $u=0$ are 0 and 4. Hence $g_0^<$ must satisfy $g_0^<(u)\sim u^4$ as $u\to 0$, and the normalization is fixed by requiring $\lim_{u\to 0} g_0^<(u)/u^4 \equiv 1$.

At zero temperature, the homogeneous solutions to the helicity zero equation are

$$g^{>}_{0,T=0}(u) = u^2 K_2(u\sqrt{q^2-\omega^2})\,, \tag{4.68a}$$

$$g^{<}_{0,T=0}(u) = \frac{8u^2}{q^2-\omega^2}\, I_2(u\sqrt{q^2-\omega^2})\,, \tag{4.68b}$$

and their Wronskian is

$$W_{0,T=0}(u) = -\frac{8u^3}{q^2-\omega^2}\,. \tag{4.69}$$

Consequently,

$$\Delta Z_0^{(4)} = \int_0^{u_h} du \left\{ \frac{g_0^{>}(u)}{W_0(u)}\, S_0(u) + J(u)\, \frac{q^2-\omega^2}{8\,\xi(u)}\, K_2\Big(\xi(u)\sqrt{q^2-\omega^2}\Big)\, S_{0,T=0}(\xi(u)) \right\}. \tag{4.70}$$

### 4. Numerics

For a given momentum $\boldsymbol{q}$, the homogeneous solutions $g_s^{>}$ are evaluated by numerically integrating the appropriate homogeneous differential equations (4.25), (4.30) and (4.34) (without sources) outward from the horizon. Then the solutions $g_s^{<}$ are evaluated by numerically integrating the same equations (without sources) inward from the boundary. Given these numerically determined homogeneous solutions, the perturbations in the helicity variables $\Delta\mathcal{T}_s$ are evaluated by numerically performing the radial integrals in Eqs. (4.64), (4.67) and (4.70), and inserting the results in Eq. (4.58).

The reconstruction of the stress-energy tensor components from the helicity variables is given in Eqs. (2.8) and (2.9). In particular, the perturbation in the energy density is

$$\Delta\mathcal{E}(t,\boldsymbol{x}) = \int \frac{d^4q}{(2\pi)^4}\, \Delta\mathcal{T}_0(\omega,\boldsymbol{q})\, e^{iQ\cdot x}\,, \tag{4.71}$$

while the perturbation in the energy flux is

$$\Delta\boldsymbol{S}(t,\boldsymbol{x}) = \int \frac{d^4q}{(2\pi)^4} \left[ \frac{\boldsymbol{q}}{q^2}\big(\omega\Delta\mathcal{T}_0 - i\mathcal{F}^0\big) + \Delta\vec{\mathcal{T}}_1(\omega,\boldsymbol{q}) \right] \times e^{iQ\cdot x}\,. \tag{4.72}$$

The frequency integrals in these expressions are trivial as the integrands are proportional to $2\pi\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})$. We write the remaining $3d$ integrals in spherical coordinates $(q,\theta,\phi)$ with $\theta$ the polar angle measured relative to the direction of the velocity $\boldsymbol{v}$. In this coordinate system we have

$$q_{\parallel} = q\cos\theta\,, \qquad q_{\perp} = q\sin\theta\,, \tag{4.73}$$

where $q_{\parallel}$ is the component of $\boldsymbol{q}$ parallel to the quark's velocity. The integration over $\phi$ is easily done analytically, as the physical problem is cylindrically symmetric about the direction $\boldsymbol{v}$. The remaining integrals over $q$ and $\theta$ are evaluated numerically.

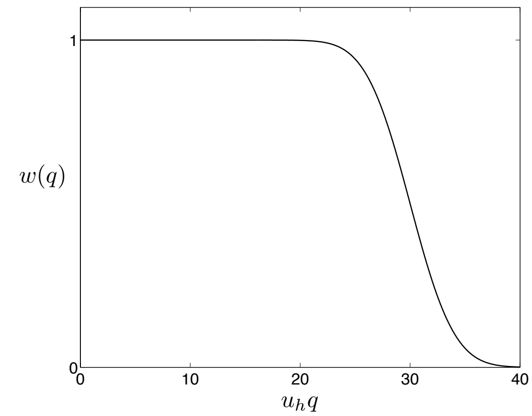


FIG. 2: A plot of the window function $w(q)$. The window function does not modify the Fourier space data for $u_h q < 20$.

The integrands (with the factor of $q^2$ from the measure) scale like $q$ times an oscillatory function in the large $q$ limit. This reflects on the fact that $\Delta T^{\mu\nu}(t,\boldsymbol{x})$ diverges like $T^2/|\boldsymbol{x}-\boldsymbol{v}t|^2$ in the vicinity of the quark. To deal with this divergence we multiply the Fourier space data by the window function

$$w(q) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{u_h q - 30}{4.5}\right)\right]. \tag{4.74}$$

This window function is shown in Fig. 2. From the figure we see that the window function does not significantly modify the Fourier space data for $u_h q < 20$. Correspondingly the window function alters the real space energy density and energy flux over length scales $\sim u_h/20$, which is much smaller than the position space grids used to make the plots shown below. With this window function, the integral over $q$ is evaluated with an upper limit of $u_h q = 40$.

## V. RESULTS

For small distances $d \equiv |\boldsymbol{x}-\boldsymbol{v}t| \ll 1/T$ away from the moving quark, the dominant contributions to the stress-energy tensor come from the $T=0$ stress-energy tensor, which scales like $1/d^4$. To highlight the medium dependent perturbations in the stress-energy tensor we defined $\Delta T^{\mu\nu}$ [in Eq. (4.56)] to be the temperature dependent perturbation to $T^{\mu\nu}$.

Figures 3–5 show plots of $\Delta\mathcal{E}(t,\boldsymbol{x})$ and $\Delta S(t,\boldsymbol{x}) \equiv |\Delta\boldsymbol{S}(t,\boldsymbol{x})|$ at quark velocities $v=1/4$, $v=3/4$ and $v=1/\sqrt{3}$, respectively.[18] The flow lines superimposed on the

[18] In making these plots, we use spatial grids with resolution $\Delta x$ equal to one to two times $1/(2\pi T)$. This limits the fidelity of

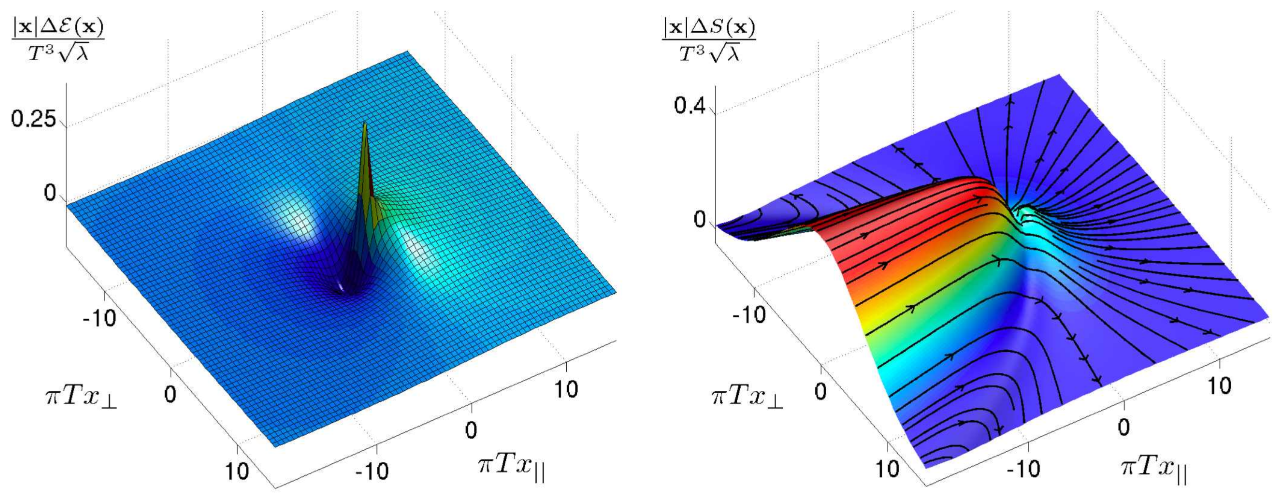


FIG. 3: Left—Position space plot of $|\boldsymbol{x}|\Delta\mathcal{E}(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = 1/4$. Right—Position space plot of $|\boldsymbol{x}|\Delta S(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = 1/4$. The flow lines on the surface are the flow lines of the energy flux $\Delta\boldsymbol{S}(\boldsymbol{x})$. There is a surplus of energy in front of the quark and a deficit behind it. Correspondingly, trailing the quark there is a stream of energy flux which moves in the same direction as the quark. Note the absence of structure in $\Delta\mathcal{E}(\boldsymbol{x})$ for distances $|\boldsymbol{x}| \gg 1/(\pi T)$.

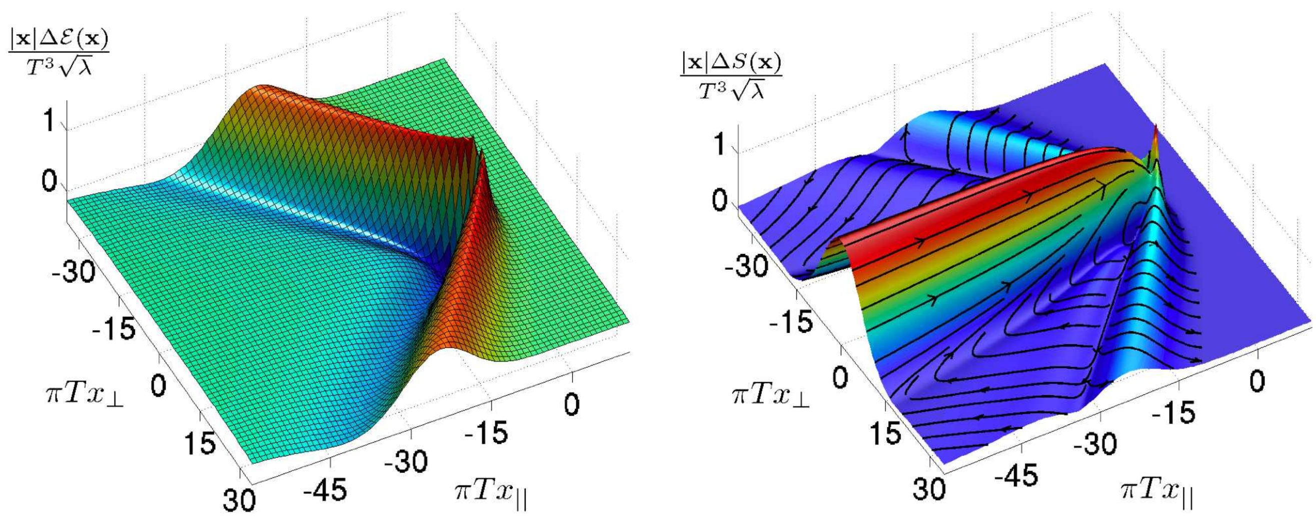


FIG. 4: Left—Plot of $|\boldsymbol{x}|\Delta\mathcal{E}(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = 3/4$. Right—Plot of $|\boldsymbol{x}|\Delta S(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = 3/4$. The flow lines on the surface are the flow lines of $\Delta\boldsymbol{S}(\boldsymbol{x})$. There is a surplus of energy in front of the quark and a deficit behind it. Correspondingly, trailing the quark there is a narrow stream of energy flux which moves in the same direction as the quark. A Mach cone, with opening half angle $\theta_M \approx 50^\circ$ is clearly visible in both the energy density and the energy flux. Near the Mach cone, the bulk of the energy flux flow is orthogonal to the wavefront.

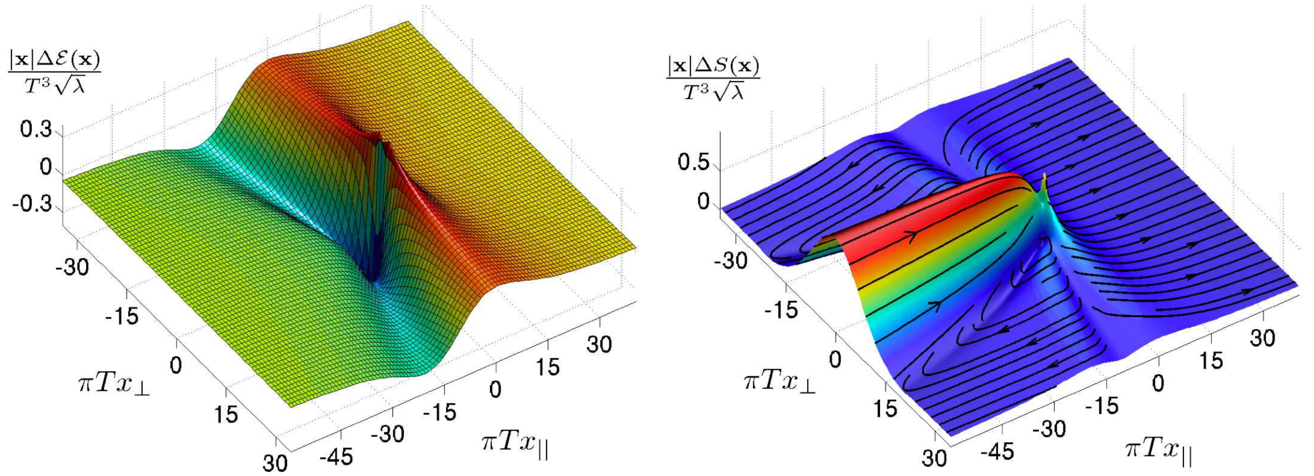


FIG. 5: Left—Plot of $|\boldsymbol{x}|\Delta\mathcal{E}(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = c_{\rm s}$. Right—Plot of $|\boldsymbol{x}|\Delta S(\boldsymbol{x})/(T^3\sqrt{\lambda})$ for $v = c_{\rm s}$. The flow lines on the surface are the flow lines of the energy flux $\Delta\boldsymbol{S}(\boldsymbol{x})$. A planar Mach cone is visible in both the energy density and the energy flux. Near the Mach cone, the bulk of the energy flux flow is orthogonal to the wavefront.

plots of $S(t,\boldsymbol{x})$ are the flow lines of $\boldsymbol{S}(t,\boldsymbol{x})$. In these plots the quark, at the time shown, is at $\boldsymbol{x}=0$. We use units in which $\pi T = 1$. Since $\mathcal{N}=4$ SYM is a conformal theory, the speed of sound is $1/\sqrt{3}$. Hence Fig. 3 shows subsonic motion, Fig. 4 shows supersonic motion, and Fig. 5 is precisely at the speed of sound.

As discussed earlier, $\Delta T^{\mu\nu}$ may be reconstructed from the helicity variables $\Delta\mathcal{T}_s$, combined with the energy-momentum conservation equation (3.1) and the vanishing trace condition of the SYM stress-energy tensor. In Appendix B we compute the small momentum limit of $\Delta\mathcal{T}_s$ to $\mathcal{O}(q^0)$ with the ratio $r \equiv \omega/q$ held fixed. Defining

$$\mathcal{K} \equiv \frac{\sqrt{\lambda}}{2\pi u_h^2\sqrt{1-v^2}}\,(2\pi)\delta(\omega - \boldsymbol{v}\cdot\boldsymbol{q}), \tag{5.1}$$

one finds[19]

$$\Delta\mathcal{T}_0 = 3\mathcal{K}\left[\frac{-ir(1+v^2)}{(1-3r^2)q} + \frac{u_h r^2(2-3r^2+v^2)}{(1-3r^2)^2}\right], \tag{5.2a}$$

$$\Delta\vec{\mathcal{T}}_1 = \mathcal{K}\,\frac{vq_\perp}{q}\left[-\frac{1}{irq} + \frac{u_h(1-4r^2)}{4r^2}\right]\hat\epsilon_1, \tag{5.2b}$$

$$\Delta\overleftrightarrow{\mathcal{T}}_2 = -\mathcal{K}\,\frac{u_h}{2}\,\frac{(vq_\perp)^2}{q^2}\,(\hat\epsilon_1\otimes\hat\epsilon_1 - \hat\epsilon_2\otimes\hat\epsilon_2), \tag{5.2c}$$

up to $\mathcal{O}(q)$ corrections.

These results for the long wavelength limit of $\Delta\mathcal{T}_s$ may be compared to the analogous long wavelength limit of the hydrodynamic quantities $\Delta\mathcal{T}_s^{\rm hydro}$ given in Eqs. (3.23). To give a meaningful comparison, we expand the hydrodynamic helicity variables in powers of $q$ with the ratio $r = \omega/q$ fixed. Writing

$$\rho = q\,\rho^{(1)} + q^2\,\rho^{(2)}\cdots, \tag{5.3a}$$

$$\boldsymbol{J}_{\rm T} = \boldsymbol{J}_{\rm T}^{(0)} + q\,\boldsymbol{J}_{\rm T}^{(1)} + \cdots, \tag{5.3b}$$

we find

$$\Delta\mathcal{T}_0^{\rm hydro} = -\frac{3\rho^{(1)}}{(1-3r^2)\,q} - \frac{9ir\,\gamma\,\rho^{(1)} + 3(1-3r^2)\rho^{(2)}}{(1-3r^2)^2}\,, \tag{5.4a}$$

$$\Delta\vec{\mathcal{T}}_1^{\rm hydro} = -\frac{\boldsymbol{J}_{\rm T}^{(0)}}{irq} + \frac{D\boldsymbol{J}_{\rm T}^{(0)} + ir\boldsymbol{J}_{\rm T}^{(1)}}{r^2}\,, \tag{5.4b}$$

$$\Delta\overleftrightarrow{\mathcal{T}}_2^{\rm hydro} = 0\,, \tag{5.4c}$$

neglecting $\mathcal{O}(q)$ corrections in Eqs. (5.4a) and (5.4b). Comparing the expansions of $\Delta\mathcal{T}_0^{\rm hydro}$ and $\Delta\vec{\mathcal{T}}_1^{\rm hydro}$ to that of $\Delta\mathcal{T}_0$ and $\Delta\vec{\mathcal{T}}_1$, we see they agree provided the transverse momentum diffusion constant and the sound attenuation constant have the expected values given earlier,[20] namely $D = 1/(4\pi T)$ and $\gamma = 1/(3\pi T)$, and the hydrodynamic sources have the expansions

$$\rho = \mathcal{K}\left[\,i\omega(1+v^2) - u_h\omega^2 + \mathcal{O}(q^3)\right], \tag{5.5a}$$

$$\boldsymbol{J}_{\rm T} = \mathcal{K}\,\frac{vq_\perp}{q}\left[1 + iu_h\omega + \mathcal{O}(q^2)\right]\hat\epsilon_1\,. \tag{5.5b}$$

---

these plots for distances $|\boldsymbol{x}| \sim \Delta x$ from the quark. As noted earlier, the temperature dependent energy density and energy flux behave like $T^2/|\Delta\boldsymbol{x}|^2$ for distances $|\Delta\boldsymbol{x}| \ll 1/T$ from the quark. See Refs. [27, 49] for a discussion of the stress-energy tensor in the near zone.

[19] These results for the small momentum limit of the stress-energy tensor may also be found in Ref. [27].

[20] Higher order transport coefficients such as $\Theta$, the coefficient multiplying second derivatives of the energy density, can be determined by carrying out the expansion of the helicity variables to higher order in $q$.

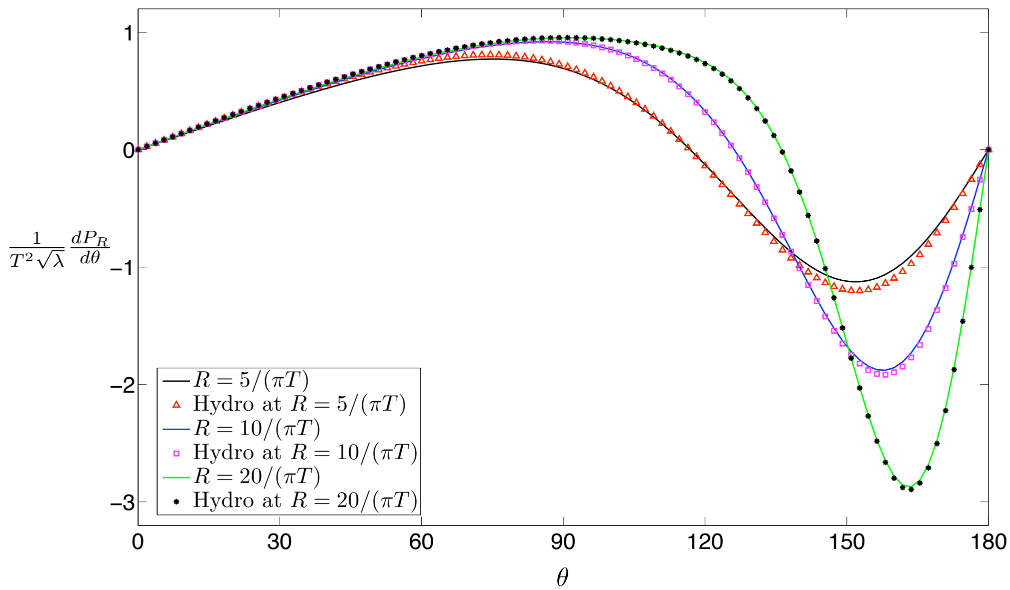


FIG. 6: Plot of $(T^2\sqrt{\lambda})^{-1}dP_R/d\theta$ for $v = 1/4$ at distances $R = 5/(\pi T)$ (black curve and red triangles), $10/(\pi T)$ (blue curve and purple squares), and $20/(\pi T)$ (green curve and black dots). Solid lines are the full AdS/CFT results, while the symbols show the linearized hydrodynamic approximation. Note the presence of energy being radiated into the forward hemisphere in front of the quark. Negative values in the backward hemisphere are due to the diffusion wake. At all three distances, the hydrodynamic approximation to $dP_R/d\theta$ agrees very well with the AdS/CFT results.

Using the fact that $J^0 \equiv F^0$, Eqs. (3.24) and (5.5a) imply that the longitudinal component of the effective source has the expansion

$$\boldsymbol{J}_{\rm L} = \mathcal{K}\,\frac{\boldsymbol{q}}{q^2}\left[\,\omega + iu_h\omega^2 - i\gamma v^2 q^2 + \mathcal{O}(q^3)\right]. \tag{5.6}$$

Combining Eqs. (5.5b) and (5.6), we find

$$\boldsymbol{J} = \mathcal{K}\left[\boldsymbol{v} + iu_h\omega\boldsymbol{v} - iv^2\gamma\boldsymbol{q} + \mathcal{O}(q^2)\right]. \tag{5.7}$$

Comparing with Eqs. (3.2) and (3.3), one sees that, as expected, the leading term in the gradient expansion of the effective source $\boldsymbol{J}$ is simply the microscopic force density $\boldsymbol{F}$. When Fourier transformed to position space, subleading corrections to the effective source generate derivatives of delta functions at the location of the quark.

As the above makes clear, the small $q$ expansions of $\Delta\mathcal{T}_0$ and $\Delta\vec{\mathcal{T}}_1$ are completely consistent with their hydrodynamic counterparts. However the small $q$ limit of $\Delta T^{\mu\nu}$, as reconstructed with the helicity variables $\Delta\mathcal{T}_s$, does not agree with the small $q$ expansion of $\Delta T^{\mu\nu}_{\rm hydro}$. There are two sources of discrepancy. First, the helicity two variable $\Delta\overleftrightarrow{\mathcal{T}}_2$ is non-vanishing at $\mathcal{O}(q^0)$ while $\Delta\overleftrightarrow{\mathcal{T}}_2^{\rm hydro}$ vanishes identically in the large $N_{\rm c}$ limit. Second, even with identical helicity variables, the reconstruction of $\Delta T^{\mu\nu}$ from its helicity variables $\Delta\mathcal{T}_s$ differs from the reconstruction of $\Delta T^{\mu\nu}_{\rm hydro}$ from $\Delta\mathcal{T}_s^{\rm hydro}$, because $\Delta T^{\mu\nu}$ and $\Delta T^{\mu\nu}_{\rm hydro}$ satisfy differing energy-momentum conservation relations. In particular, $\Delta T^{\mu\nu}$ satisfies the exact microscopic conservation relation (3.1) with the force density $F^\nu$ on the right hand side, while $\Delta T^{\mu\nu}_{\rm hydro}$ satisfies the effective energy-momentum conservation relation (3.12) involving the effective source $J^\nu$. However, examining the small $q$ limits of $\Delta T^{\mu\nu}$ and $\Delta T^{\mu\nu}_{\rm hydro}$, we find

$$\Delta T^{\mu\nu} = \Delta T^{\mu\mu}_{\rm hydro} + \mathcal{A}^{\mu\nu}\,, \tag{5.8}$$

where

$$\mathcal{A}^{0\mu} = 0\,, \tag{5.9a}$$

$$\mathcal{A}^{ij} = \mathcal{K}\left(v^2\gamma\,\delta_{ij} - u_h\,v_iv_j\right). \tag{5.9b}$$

When Fourier transformed back to position space, $\mathcal{A}^{\mu\nu}$ has point support at the location of the quark and satisfies

$$\partial_\mu\mathcal{A}^{\mu\nu} = F^\mu - J^\mu\,. \tag{5.10}$$

In other words, $\mathcal{A}^{\mu\nu}$ is a local contribution to the stress-energy tensor which is precisely tailored to compensate for the difference between the effective source $J^\mu$ and the microscopic force density. This is a necessary result. The gradient expansions of $\Delta T^{\mu\nu}$ and $\Delta T^{\mu\nu}_{\rm hydro}$ must agree far from the quark. However, in the long wavelength limit, any discrepancy between $\Delta T^{\mu\nu}$ and $\Delta T^{\mu\nu}_{\rm hydro}$ in the near zone may be represented as a term with point support at the location of the quark. Because $\Delta T^{\mu\nu}$ and $\Delta T^{\mu\nu}_{\rm hydro}$ satisfy different conservation equations, $\mathcal{A}^{\mu\nu}$ must obey Eq. (5.10) to all orders in the gradient expansion of $J^\mu$. It is, of course, reassuring to see $\mathcal{A}^{\mu\nu}$ naturally emerge from the gravitational calculation. Note that information from all three gauge invariants $Z_s$ is used to calculate $\mathcal{A}^{\mu\nu}$, thus showing the non-trivial interplay between

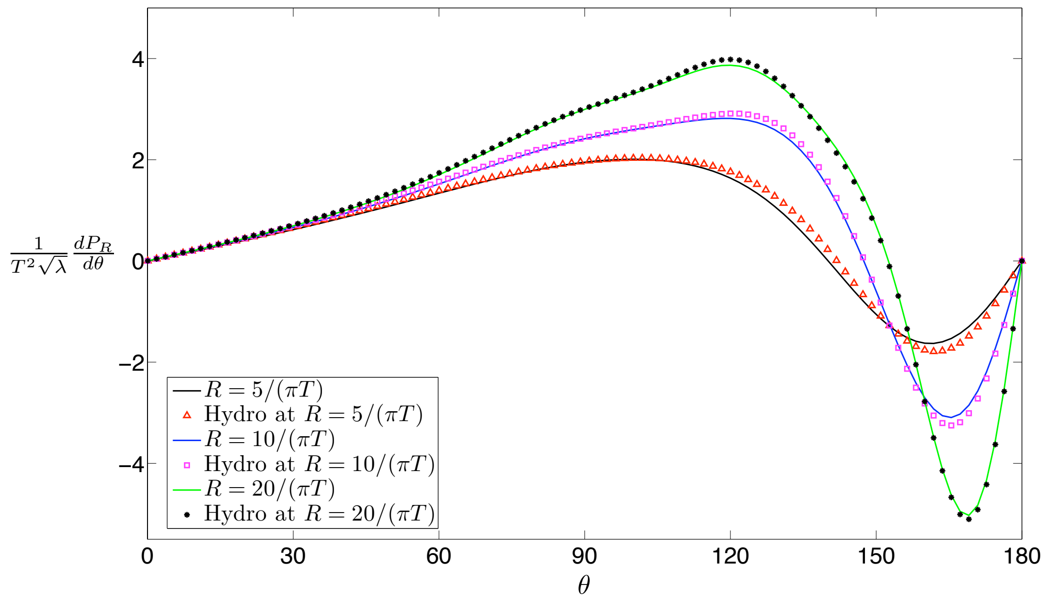


FIG. 7: Plot of $(T^2\sqrt{\lambda})^{-1}dP_R/d\theta$ at $v = c_{\rm s}$ and distances $R = 5/(\pi T)$, $10/(\pi T)$, and $20/(\pi T)$. The labeling of curves is the same as in Fig. 6. Note the presence of energy being radiated into the forward hemisphere in front of the quark, together with the inward flux associated with the diffusion wake in the backward hemisphere. The Mach cone is centered at $\theta = 90°$. At all three distances, the hydrodynamic approximation to $dP_R/d\theta$ agrees very well with the AdS/CFT results.

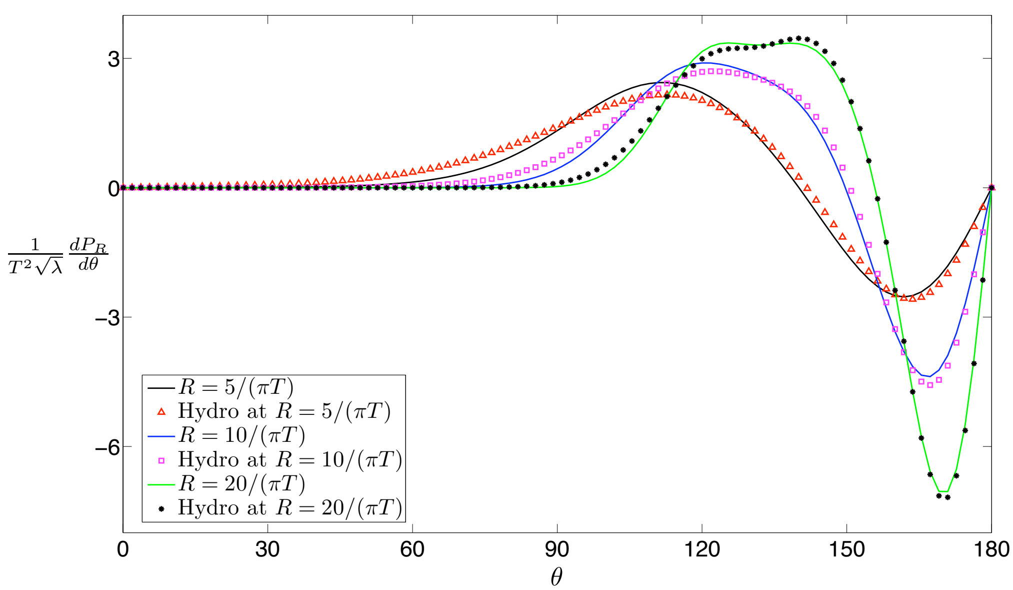


FIG. 8: Plot of $(T^2\sqrt{\lambda})^{-1}dP_R/d\theta$ at $v = 3/4$ and distances $R = 5/(\pi T)$, $10/(\pi T)$, and $20/(\pi T)$. The labeling of curves is the same as in Fig. 6. Note the absence of energy radiated in front of the quark, as the quark is moving supersonically. The Mach cone near 130°is sharper than in the transonic case shown in Fig. 7. The hydrodynamic approximation to $dP_R/d\theta$ improves as $R$ grows, but is already fairly good at $R = 5/(\pi T)$.

the decoupled gravitational equations of motion (4.25), (4.30), and (4.34).

It is illuminating to compare hydrodynamics to the AdS/CFT results in position space. A simple quantity to compare, which involves both the sound and diffusion modes, is the instantaneous angular distribution of the energy flux at a distance $R$ from the quark. Choosing to evaluate this at $t = 0$, when the quark is at the origin, the angular distribution of the energy flux is

$$\frac{dP_R}{d\Omega} \equiv R^2\,\hat{x}\cdot\Delta\boldsymbol{S}(0, R\,\hat{x})\,. \tag{5.11}$$

Because of the rotational symmetry about the velocity axis, its sufficient to consider the energy flux per unit polar angle $\theta$,

$$\frac{dP_R}{d\theta} = 2\pi R^2\,\sin\theta\left(\cos\theta\,\Delta S_{\parallel} + \sin\theta\,\Delta S_{\perp}\right), \tag{5.12}$$

with $\theta = 0$ corresponding to the $\hat{v}$ axis and $\Delta S_{\parallel}$ and $\Delta S_{\perp}$ the components of the energy flux parallel and perpendicular to the velocity vector $\boldsymbol{v}$, respectively. We plot $dP_R/d\theta$ for $v = 1/4$ in Fig. 6, $v = c_{\rm s}$ in Fig. 7, and $v = 3/4$ in Fig. 8. In all three figures we plot the angular distribution of the flux at distances $R = 5/(\pi T)$, $10/(\pi T)$, and $20/(\pi T)$. The hydrodynamics curves were made by numerically integrating Eqs. (3.27) and (3.28) with the leading order effective source $J^\mu = F^\mu$. Hydrodynamics becomes increasingly accurate at longer distances, but even at $R = 5/(\pi T)$ one sees rather good agreement between linearized hydrodynamics and the full AdS/CFT results.

We emphasize that $dP_R/d\Omega$ is not the same as the power radiated in the rest frame of the quark. In the quark's rest frame, the total power radiated through spheres of radius $R$ is independent of $R$. However, in the rest frame of the plasma, the total flux, $\int d\Omega\,(dP_R/d\Omega)$, through a sphere centered on the instantaneous position of the quark grows with increasing $R$ as successively larger spheres capture energy radiated ever farther back in time.

## VI. DISCUSSION

Much of the qualitative and quantitative structure in the plots of $\Delta\mathcal{E}(t,\boldsymbol{x})$ and $\Delta S(t,\boldsymbol{x})$ can be understood from hydrodynamic considerations alone. This statement is reinforced by Eq. (5.8), which shows that the long wavelength limit of the stress tensor as computed with gauge/string duality coincides with the linearized hydrodynamics result, provided the latter is computed with the correct effective source. As discussed in Section III, long wavelength perturbations in the energy density satisfy the diffusive wave equation (3.29) describing sound waves with the dispersion relation

$$\omega \approx \pm c_{\rm s}\,q - \tfrac{i}{2}\,\gamma\,q^2\,, \tag{6.1}$$

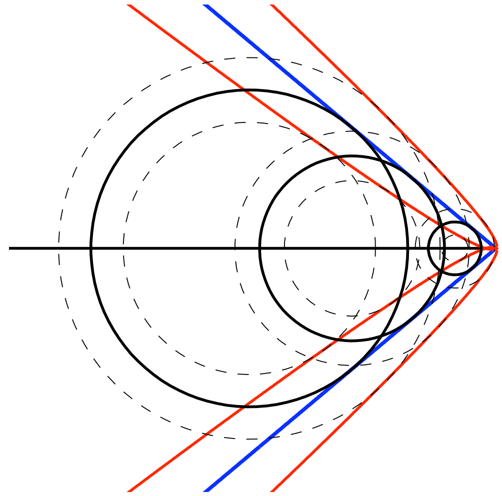

FIG. 9: A schematic representation of the Green's function solution to the sound equations (3.29) and (3.35) for supersonic motion. As the quark moves, it creates sound disturbances which propagate out in spherical shells. The resulting sound waves add coherently along the Mach cone, shown in blue. Because the sound waves are damped, each spherical wave will broaden as it propagates. The broadening is schematically indicated by the dashed lines. This implies that the Mach cone broadens with increasing distance from the quark, as indicated by the red lines.

[with $c_{\rm s} = 1/\sqrt{3}$ and $\gamma = 1/(3\pi T)$]. A textbook constructive interference argument, illustrated in Fig. 9, shows that a projectile moving supersonically will produce a Mach cone with an opening half-angle given by $\sin\theta_{\rm M} = c_{\rm s}/v$ (where $\tan\theta_{\rm M} \equiv -x_\perp/x_\parallel$). For $v = 3/4$ this is $\theta_{\rm M} = 50.3°$. As is evident in Fig. 4, at $v = 3/4$ the energy wake is concentrated, as expected, along a 50° cone with the the bulk of the associated energy flux flowing perpendicular to the wave front. Similarly, in Fig. 5, at $v = c_{\rm s}$ the energy wake is concentrated along the plane $x_\parallel = 0$ with the bulk of the associated energy flux again flowing perpendicular to the wave front.

The damping of sound waves implies that their waveforms must broaden as they propagate. This in turn implies that the Mach cone broadens with increasing distance from the quark. This behavior is clearly seen in Fig. 4. As one can reason from Fig. 9, the width $\Gamma_{\rm Mach}$ of the Mach cone, defined as the length scale over which the energy density attenuates (exponentially) in the forward direction, increases with distance $d$ from the quark like

$$\Gamma_{\rm Mach} \sim \frac{(\gamma\,d)^{1/2}}{(v^2 - c_{\rm s}^2)^{1/4}}\,. \tag{6.2}$$

This attenuation length diverges as $v \to c_{\rm s}$; in this limit there is only power law falloff of the Mach cone in front of the quark. This reflects the fact that at $v = c_{\rm s}$, sound waves emitted by the quark arbitrarily far in the past can add coherently to the Mach cone at finite distances from the quark. If a given sound wave has to travel a

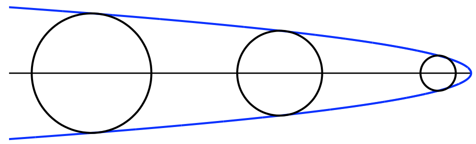

FIG. 10: A schematic representation of the Greens function solution to the diffusion equation (3.37). As the quark moves, momentum deposited in the diffusion channel at time $t_0$ diffuses over the length scale $\sqrt{D(t-t_0)}$. This results in the broadening of the diffusion wake with increasing distance from the quark.

distance $\ell$ to reach the Mach cone, the width of the wave will have broadened to $\sim \sqrt{\gamma\ell/c_s}$. At precisely $v = c_s$, sound waves with $\ell \to \infty$ contribute to the Mach cone at finite distances from the quark, leading to power-law falloff of the Mach cone.

From Eq. (3.37), we see that part of the long wavelength perturbation in the energy flux satisfies a diffusion equation with diffusion constant $D = 1/(4\pi T)$. Fig. 10 shows a pictorial representation of the the Greens function solution to the diffusion equation (3.37). As the quark moves, momentum transferred to the diffusion mode at time $t$ mode will gradually diffuse outward, resulting in a stream of momentum trailing the quark. The width of the corresponding *diffusion wake* grows with distance $d$ from the quark like

$$\Gamma_{\rm diffusion} \sim \sqrt{Dd/v}. \tag{6.3}$$

This behavior is evident in the energy flux plots in Figs. 3–5. The directional dependence of the diffusion wake is dictated (in the long wavelength limit) by the directional dependence of the effective source $\boldsymbol{J}_{\rm diffusion}$, (defined in Eq. 3.38). To leading order in gradients, $\boldsymbol{J}_{\rm diffusion} \propto \boldsymbol{v}$, so the diffusion wake should flow in the same direction as the quark's motion. Again, this behavior is clearly seen in the energy flux plots of Figs. 3–5.

It is noteworthy that the diffusion wake is not seen at all in the plots of the energy density. This can be understood from hydrodynamic and large $N_c$ scaling considerations. First of all, note that the *kinetic energy* associated with bulk motion of the fluid is negligible in the large $N_c$ limit. This follows from the fact that the fluid velocity which results from the motion of the quark through the plasma is $\mathcal{O}(1/N_c^2)$. (The fluid velocity $\boldsymbol{u}$ equals the $\mathcal{O}(N_c^0)$ momentum density divided by the $\mathcal{O}(N_c^2)$ enthalpy, $\epsilon{+}p$.) And hence the associated bulk kinetic energy density, given by $(\epsilon{+}p)\,\boldsymbol{u}^2$, is also $\mathcal{O}(1/N_c^2)$.[21] This means that an $\mathcal{O}(N_c^0)$ momentum density need not produce any imprint on the $\mathcal{O}(N_c^0)$ energy density perturbation, exactly as seen in Figs. 3–5. To see this more formally note, from Eq. (3.34b), that the divergence of the diffusive energy flux has point support at the quark. That is, $\nabla \cdot \boldsymbol{S}_{\rm diffusion} = 0$ everywhere away from the quark. This implies that the momentum transported via the diffusion flux does not involve any corresponding perturbation in the energy density. This is analogous, in electromagnetism, to the possibility of having a transverse electric current density without any corresponding charge density.

To put our discussion on a somewhat more quantitative footing, we now turn to the plots of $dP_R/d\theta$ shown in Figs. 6–8. For subsonic and transonic motion (Figs. 6 and 7), $dP_R/d\theta$ is reproduced quite well by hydrodynamics all the way down to distances $R = 5/(\pi T)$ from the quark. For supersonic motion (Fig. 8) the agreement between hydrodynamics and the complete result is not quite as good, quantitatively, at $R = 5/(\pi T)$, but the level of agreement is still rather remarkable. The region of largest discrepancy occurs on or near the Mach cone. It is natural that the hydrodynamic approximation to the stress tensor deviates more for supersonic motion than for subsonic motion. This is because gradients in the energy density and energy flux are largest on the Mach cone. This behavior is clearly seen in the near zone behavior of the energy density in Fig. 4. However, in constructing the hydrodynamic approximation to the stress tensor in Eq. (3.11), we have only kept the leading viscous terms in the gradient expansion and neglected $\Theta$ and all other higher order terms in the gradient expansion of the stress.

We emphasize the importance of including the nonzero viscosity when evaluating the hydrodynamic curves shown in Figs. 6–8. Had we neglected viscosity, the stress tensor would be discontinuous on the Mach cone and, for $v = c_s$, would actually diverge on the Mach cone. Furthermore, in the limit of zero viscosity the width of the diffusion wake vanishes, leaving a diffusion wake proportional to $\delta^2(\boldsymbol{x}_\perp)\Theta(-x_\parallel + vt)$. Therefore, neglecting viscosity yields a very poor approximation to the stress tensor in the vicinity of both the Mach cone and the diffusion wake.

The agreement between hydrodynamics and AdS/CFT at distances $d \gtrsim 1/T$ from the quark should be contrasted with the corresponding situation at weak coupling. If the 't Hooft coupling $\lambda \ll 1$, then the SYM plasma has a quasi-particle description. The mean free path of quasiparticles (gluons, fermions, or scalars) scales like [33]

$$\ell_{\rm mfp} \sim \frac{1}{T\lambda^2 \ln \lambda^{-1}}\,. \tag{6.4}$$

Therefore, the mean free path, at weak coupling, is parametrically longer than $1/T$. Hydrodynamics is only valid on spatial scales large compared to the mean free path. So in contrast to the strong coupling results discussed above, in a weakly coupled plasma hydrodynamics is never valid on distance scales comparable to $1/T$.

It is interesting to ask how the energy and momentum lost by the quark is distributed into the sound and diffusion modes. This question has been addressed in the

[21] Consequently, in the large $N_c$ limit, all of the energy lost by a quark traversing the plasma is turned into heat.

steady state limit in Refs [31, 32]. However, for reasons which are outlined below, we question the physical value of addressing this topic in the limit where the quark has been moving forever.

Let $V$ denote a large sphere of radius $R$ whose location is fixed and centered on the position of the quark at time $t=0$. The rate of change of the energy inside volume $V$ is

$$\frac{dE_V}{dt} = \int_V d^3x\, \partial_0 T^{00}(t,\boldsymbol{x})\,. \tag{6.5}$$

After using the energy-momentum conservation equation (3.1), and separating the energy flux into sound and diffusive pieces, Eq. (6.5) becomes

$$\frac{dE_V}{dt} = f^0 - \int_{\partial V} d\Sigma \cdot \boldsymbol{S}_{\rm sound} - \int_{\partial V} d\Sigma \cdot \boldsymbol{S}_{\rm diffusion}\,. \tag{6.6}$$

The first term on the right is simply the rate at which the quark deposits energy into the plasma while the second and third terms are the rates at which energy is added to and removed from $V$ via the sound and diffusion fluxes.

In the steady state limit, it is easy to calculate the rate at which energy is added to $V$ by the diffusion flux. If $R \gg 1/T$, then one may compute the surface integrals in Eq. (6.6) using hydrodynamics. The long wavelength limit of the diffusion flux satisfies[22]

$$\nabla \cdot \boldsymbol{S}_{\rm diffusion} = -\frac{1}{v^2}\, F^0\,. \tag{6.7}$$

From this expression we see that the diffusion flux *adds* energy to the volume $V$ at the rate

$$-\int_{\partial V} d\Sigma \cdot \boldsymbol{S}_{\rm diffusion} = \frac{f^0}{v^2}\,. \tag{6.8}$$

The source of the energy supplied to $V$ via the diffusion flux comes from the quark itself — in the distant past when the quark was outside of the volume $V$, it created the diffusion flux which subsequently flows into $V$. (The fact that the diffusion flux adds energy to $V$ simply reflects the fact that the diffusion flux flows in the same direction as the quark's velocity.) Note that the rate at which energy is added to $V$ via the diffusion flux is independent of the size of $V$, and so is non-vanishing in the $R \to \infty$ limit. Furthermore the rate (6.8) has a finite, non-zero limit as $v \to 0$ [recall that $f^0 \equiv \boldsymbol{v} \cdot \boldsymbol{f} = \mathcal{O}(v^2)$]. So the diffusion flux adds energy to $V$ even when the quark's velocity is taken to be arbitrarily small! The origin of this peculiar behavior comes from the assumption that the quark has been moving forever. In this limit, diffusive energy flux deposited in the arbitrarily distant past can influence the rate of change of the energy in the volume $V$. Completely analogous conclusions also hold for the sound flux.[23]

The assumption that the quark has been moving at constant velocity forever presents a great technical simplification in both the gravitational and hydrodynamic computation of the quark wake. However, the utility of this simplifying assumption is limited to questions which are insensitive to the details of the quark's trajectory in the distant past. Asking how the energy and momentum lost by the quark are distributed into the sound and diffusion modes, when the quark has been moving forever, is not such a question. A better and still analytically tractable question is to consider is how much energy and momentum are deposited into each mode when the quark has been moving at constant velocity $v$ for a long but finite period of time $\Delta t = t_f - t_i$. The total energy transfered to the plasma is then

$$\Delta E(t) = \int d^3x\, T^{00}(t,\boldsymbol{x})\,, \tag{6.9}$$

where the integration is taken over all of space. For late times after the quark's motion has ceased, all of the energy deposited by the quark will be transported away via sound waves. That is, because the quark has been moving for a finite period of time, no fluxes at spatial infinity contribute to $\Delta E(t)$.

The total momentum deposited by the quark is

$$\Delta\boldsymbol{p} = \Delta t\, \boldsymbol{f}\,. \tag{6.10}$$

To see what fraction of the quark's momentum is deposited in each mode, we will examine the diffusion mode and then infer the momentum deposited in the sound mode via momentum conservation. The total momentum transfered to the diffusion mode is

$$\Delta\boldsymbol{p}_{\rm diffusion}(t) = \int d^3x\, \boldsymbol{S}_{\rm diffusion}(t,\boldsymbol{x})\,. \tag{6.11}$$

If the interval during which the quark has been moving is sufficiently long, then the total momentum transfered to the diffusion mode may be computed with hydrodynamics. Moreover, in the limit $\Delta t \gg 1/T$, the vast majority of the volume of the diffusion wake will be well approximated by its steady state limit. In the steady state limit it is easy to solve the diffusion equation (3.37) with

[22] This equation was first obtained in Refs. [31, 32] via a gravitational calculation. However, it follows directly from linearized hydrodynamics using the leading order effective source $J^\mu = F^\mu$. Specifically, Eq. (6.7) follows from the definition of the diffusion flux in Eq. (3.34b) plus the observation that $\nabla \cdot \boldsymbol{S}_{\rm nonlocal} = -F^0/v^2$, to leading order in gradients.

[23] To demonstrate this, one may solve the wave equation (3.35) in the analytically tractable limit of small velocity. When $v \ll c_{\rm s}$ no Mach cone is generated and the effects of viscosity on the long distance behavior of the sound energy flux are negligible. Taking the duration of time in which the quark has been moving $\Delta t \to \infty$ before taking $v \to 0$, one finds that the rate at which the sound flux removes energy from the volume $V$ has a finite, non-zero limiting value.

the corresponding leading order effective source given in Eq. (3.39b). Assuming that the quark is at $\boldsymbol{x} = 0$ at time $t_f$, the result reads

$$\boldsymbol{S}_{\rm diffusion}(t{=}t_f, \boldsymbol{x}) = \frac{\boldsymbol{f}}{4\pi D r} \, \exp\left[-\frac{v}{2D}\left(x_\parallel + r\right)\right] + \cdots \,, \tag{6.12}$$

where $r = |\boldsymbol{x}|$ and the ellipsis denotes corrections suppressed by inverse powers of $r$ (coming from higher order gradient corrections). Eq. (6.12) provides a good approximation to the diffusion wake everywhere except in a region of size $\sim \sqrt{D\Delta t}$ centered about the quark's starting point. In the $\Delta t \to \infty$ limit this region is negligible compared to the *length* of the diffusion wake, which is $v\,\Delta t$. To obtain the total momentum transfered to the diffusion wake, we integrate Eq. (6.12) over a length $v\,\Delta t$ in the $x_\parallel$ direction. Specifically, we have

$$\Delta\boldsymbol{p}_{\rm diffusion}(t_f) = \int_{-v\Delta t}^{0} dx_\parallel \int d^2x_\perp \; \boldsymbol{S}_{\rm diffusion}(t_f, \boldsymbol{x}) \,. \tag{6.13}$$

The integral is elementary to carry out in the $\Delta t \to \infty$ limit. The result reads

$$\Delta\boldsymbol{p}_{\rm diffusion}(t_f) = \Delta t \, \boldsymbol{f}. \tag{6.14}$$

This shows that in the $\Delta t \to \infty$ limit all of the momentum lost by the quark is transfered to the diffusion mode. Correspondingly, the sound mode carries no net momentum. The qualitative origin of this statement can be understood by inspecting the momentum density near the Mach cone in Figs. 4–5. In particular, the momentum density (*i.e.*, energy flux) flows predominately parallel and anti-parallel to the Mach cone normal. Evidently, when integrated over all space, the opposing flows cancel.

The direct applicability to RHIC physics of results obtained from studying the wake produced by a quark moving for an unboundedly long time through a plasma of infinite extent is, of course, questionable at best. The plasma produced at RHIC is very dynamic. The expansion of the plasma and the resulting variable speed of sound can have a dramatic effect on the structure of the quark wake [24]. However, the agreement between hydrodynamics and gauge/string duality in describing the structure of the quark wake in strongly coupled SYM does have relevance for RHIC. In particular, at least for the particular problem addressed in this paper, it shows that one can use hydrodynamics to address physics at length scales all the way down to distances less than two times $1/T$.

Because the quark-gluon plasma produced in heavy ion collisions is believed to be strongly coupled [1, 2], the agreement between hydrodynamics and gauge/string duality in a strongly coupled SYM plasma bolsters the notion that one should be able to model accurately the wake produced by a high energy quark (or gluon) moving through a QCD plasma merely using hydrodynamics. Of course, the application of hydrodynamics requires among other things, knowledge of the viscosity of the plasma and knowledge of the drag force on the quark, neither of which are under very good control for real quark-gluon plasma at accessible temperatures. (And, as noted earlier, neglecting viscosity will yield a poor approximation in the vicinity of both the Mach cone and the diffusion wake.) However, at least when $N_{\rm c}$ is large (and the hydrodynamic equations are linear), the *structure* of quark's wake is rather insensitive to magnitude of the drag force. For example, in the steady state limit the magnitude of the drag force enters only as an overall normalization of the quark wake. As a next step toward a more realistic treatment, it would be interesting, and should be feasible, to study the wake produced by a quark traversing an expanding and cooling $\mathcal{N}{=}4$ SYM plasma [16].

## VII. CONCLUSIONS

Using gauge/string duality, we have evaluated the perturbation in the stress-energy tensor due to the presence of a fundamental quark moving through a strongly coupled large $N_{\rm c}$ maximally supersymmetric Yang-Mills plasma. Our plots of the energy density and energy flux in Figs. 3–5 clearly display the formation of Mach cone for velocities $v \geq c_{\rm s}$, together with the presence of a diffusive wake in the energy flux at all velocities.

We have argued that the effective source for hydrodynamics is, to leading order in gradients, simply determined by the microscopic drag force acting on the quark. By comparing the small momentum asymptotics of the stress-energy tensor with the predictions of hydrodynamics, we were able to determine the first subleading correction to the effective hydrodynamic source.

Using the leading order effective source for hydrodynamics, we compared the hydrodynamic prediction to the complete result obtained via gauge/string duality for the instantaneous angular distribution of power radiated through spherical shells a distance $R$ from the quark. The comparison showed remarkably good agreement between hydrodynamics and the exact result at distances all the way down to $R \sim 1.6/T$ from the quark. This reinforces the notion that wakes produced by high energy particles traversing a real quark-gluon plasma can be well modeled merely using hydrodynamics.

### Acknowledgments

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## APPENDIX A: THE BOUNDARY ACTION

The total gravitational action for the heavy quark effective theory is given by

$$S_{\rm G} = S_{\rm EH} + S_{\rm GH} + S_{\rm DBI} + S_{\rm NG} + S_{\rm CT}\,, \tag{A1}$$

where

$$S_{\rm EH} \equiv \frac{1}{2\kappa_5^2}\int d^5x\,\sqrt{-G}\,(R{+}2\Lambda)\,, \tag{A2a}$$

$$S_{\rm GH} \equiv \frac{1}{2\kappa_5^2}\int d^4x\,\sqrt{-\gamma}\;2K\,, \tag{A2b}$$

and the other pieces were described in section IV. We write the metric in the bulk as

$$G_{MN} = G^{(0)}_{MN} + h_{MN}\,, \tag{A3}$$

and choose to work in a gauge where $h_{5M} \equiv 0$. We similarly write the metric on the boundary as

$$\gamma_{\mu\nu} = \gamma^{(0)}_{\mu\nu} + h_{\mu\nu}\,. \tag{A4}$$

To quadratic order in $h_{\mu\nu}$ we have

$$\sqrt{-G}\,(R+2\Lambda) = \sqrt{-G_{(0)}}\left(D_P W^P + \mathcal{L}_0 - \frac{2d}{L^2}\right), \tag{A5}$$

where

$$\begin{aligned}\mathcal{L}_0 = {} & \tfrac14 D_M h D^M h + \tfrac12 D_P h_{MN} D^M h^{NP} \\ & - \tfrac12 D_P h^{MP} D_M h - \tfrac14 D_P h^{MN} D^P h_{MN} \\ & + \frac{d}{2L^2}\left(\tfrac12 h^2 - h^{MN} h_{MN}\right),\end{aligned} \tag{A6}$$

and

$$\begin{aligned}W^P = {} & -\tfrac12 h D^P h + \tfrac12 D_M\left(h h^{MP}\right) \\ & + \tfrac12 h^{PM} D_M h + h^{MN} D^P h_{MN} \\ & - D^M\left(h^{PN} h_{MN}\right) - D^P h + D^M h^P_{\ M}\,.\end{aligned} \tag{A7}$$

To vary the gravitational action, let

$$h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu}\,, \tag{A8}$$

where $h_{\mu\nu}$ satisfies the linearized equations of motion (4.18) and $\delta h_{\mu\nu}$ is infinitesimal. The variation of the bulk action density is

$$\sqrt{-G_{(0)}}\,\delta\mathcal{L}_0 = \sqrt{-G_{(0)}}\left(D_P\,\delta V^P - \kappa_5^2\,\delta h_{MN}\,t^{MN}\right), \tag{A9}$$

where

$$\begin{aligned}\delta V^P = {} & \tfrac12 \delta h\, D^P h - \tfrac12 \delta h_{MN}\, D^P h^{MN} + \delta h_{MN}\, D^M h^{PN} \\ & - \tfrac12 \delta h\, D_N h^{PN} - \tfrac12 \delta h^{PN}\, D_N h\,,\end{aligned} \tag{A10}$$

and $t^{MN}$ is the bulk stress-energy tensor, whose only source (in the limit of large quark mass) is the string hanging down to the horizon. The variation of the Nambu-Goto action is, by definition

$$\delta S_{\rm NG} = \int d^5x \sqrt{-G_{(0)}}\;\tfrac12 \delta h_{MN} t^{MN}\,. \tag{A11}$$

Hence the variation in the Nambu-Goto action cancels the last term in Eq. (A9). As discussed in section IV, in the large quark mass limit, the DBI action may be replaced with the ordinary Maxwell action for the electromagnetic field (residing on the boundary) which accelerates the quark.

The trace of the extrinsic curvature is given by

$$K = \nabla_M n^M \tag{A12}$$

where $n^M = -\sqrt{G^{55}}\,\delta^{5M}$ is the outward pointing normal to the boundary and $\nabla_\mu$ denotes covariant differentiation with respect to the full metric $G_{\mu\nu}$. Using

$$\sqrt{\gamma}\,\nabla_M n^M = -\sqrt{G^{55}}\,\partial_u\sqrt{\gamma}, \tag{A13}$$

we can write

$$S_{\rm GH} = -\frac{1}{\kappa_5^2}\lim_{u\to0}\sqrt{G^{55}}\,\partial_u\int d^4x\,\sqrt{-\gamma}\,. \tag{A14}$$

Assembling the pieces, and specializing to the AdS-Schwarzschild bulk geometry, we have

$$\delta S_{\rm G} = \delta S_{\rm B} + \delta S_{\rm horizon}\,, \tag{A15}$$

where

$$\begin{aligned}\delta S_{\rm B} = {} & \frac{1}{2\kappa_5^2}\lim_{u\to0}\int d^4x\Big[-\frac{L^3 f}{2u^3}\,\delta h\,\partial_u h + \frac{L^3 f}{2u^3}\,\delta h^\mu_{\ \nu}\,\partial_u h^\nu_{\ \mu} \\ & + \Big\{\chi\,\delta h_{\mu\nu}\,h^{\mu\nu} - \frac{\chi}{2}\,\delta h\,h - \frac{Lf'}{4fu}\,\delta h_{00}\,h - \frac{L^3 f'}{2u^3}\,\delta h_{0i}\,h^{0i}\Big\} \\ & - \frac{3\left(\sqrt{f}-1\right)L}{\sqrt{f}\,u^2}\,\delta h_{00} - \chi\,\delta h^i_{\ i}\Big] + \delta S_{\rm EM}\,,\end{aligned} \tag{A16}$$

with

$$\chi \equiv \frac{L^3\left(-6f + 6\sqrt{f} + uf'\right)}{2u^4}\,, \tag{A17}$$

and $\delta S_{\rm horizon}$ is a surface term at the black-brane horizon whose explicit form will not be needed. From the linearized field equations (4.18) it is easy to see that near the boundary $h_{\mu\nu} \sim u^2$. Using $\delta h_{\mu\nu} \equiv \frac{L^2}{u^2}\,\delta H_{\mu\nu}$, we see that all of the terms in the curly braces in Eq. (A16) scale like $u^4\,\delta H_{\mu\nu}$ as $u \to 0$. It follows that these terms will not contribute to $\delta S_{\rm G}/\delta H_{\mu\nu}(x,u)$ in the $u \to 0$ limit. Neglecting terms which vanish in the $u \to 0$ limit we therefore have

$$\begin{aligned}\delta S_{\rm B} = \lim_{u\to0}\int d^4x\Big[ & \frac{L^3}{4\kappa_5^2 u^3}\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right) \\ & \times\delta H_{\alpha\beta}\,\partial_u H_{\mu\nu} + \tfrac12 \delta H_{\mu\nu}\,T^{\mu\nu}_{\rm eq}\Big] + \delta S_{\rm EM}\,,\end{aligned} \tag{A18}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski space metric and

$$T^{\mu\nu}_{\rm eq} = \epsilon\, \mathrm{diag}(1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \tag{A19}$$

is the stress-energy tensor of the equilibrium $\mathcal{N}=4$ SYM plasma. The explicit value of the equilibrium energy density is

$$\epsilon \equiv \frac{3L^3}{2\kappa_5^2 u_h^4} = \tfrac{3}{8} N_{\rm c}^2 \pi^2 T^4 \,. \tag{A20}$$

We write the variation of the boundary action in Fourier space and wish to express it in terms of the gauge invariants $Z_s$. The variation in the boundary action is

$$\delta S_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4} \Bigg[ \frac{L^3}{4\kappa_5^2 u^3} \left( \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} \right) \times \delta\mathcal{H}_{\alpha\beta}\, \partial_u \mathcal{H}_{\mu\nu} + \tfrac{1}{2} \delta\mathcal{H}_{\mu\nu}\, T^{\mu\nu}_{\rm eq} \Bigg] + \delta S_{\rm EM} \,. \tag{A21}$$

Here products are understood to be evaluated at antipodal momentum as dictated by the reality condition $\mathcal{H}^*_{\mu\nu}(\omega, \boldsymbol{q}) = \mathcal{H}_{\mu\nu}(-\boldsymbol{q}, -\omega)$.

### 1. Tensor Mode

The terms in Eq. (A21) constructed out of radial derivatives of $Z_2^{ab} \equiv \mathcal{H}_{ab} - \frac{1}{2}\delta_{ab}\mathcal{H}$ are

$$S^2_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4}\, \mathcal{A}_2\, \delta Z_2^{ab}\, \partial_u Z_2^{ab} \,, \tag{A22}$$

where

$$\mathcal{A}_2 = \frac{L^3}{4u^3\kappa_5^2} \,. \tag{A23}$$

### 2. Vector Mode

The terms in Eq. (A21) constructed out of radial derivatives of $\mathcal{H}_{aq}$ and $\mathcal{H}_{0a}$ are

$$\delta S^1_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4}\, \frac{L^3}{2\kappa_5^2 u^3} \left[ \delta\mathcal{H}_{0a}\, \partial_u \mathcal{H}_{0a} + \delta\mathcal{H}_{qa}\, \partial_u \mathcal{H}_{qa} \right] . \tag{A24}$$

We now use Eq. (4.29) to eliminate $\partial_u \mathcal{H}_{qa}$. Doing so, we find

$$\delta S^1_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4} \left[ \mathcal{A}_1 \left( q\delta\mathcal{H}_{0a} + \omega\delta\mathcal{H}_{aq} \right) Z_1^a + \delta\mathcal{H}_{aq}\, \mathcal{J}^{aq} \right] , \tag{A25}$$

where

$$\mathcal{A}_1 = -\frac{L^3}{2q\kappa_5^2 u^3} \,, \tag{A26}$$

and

$$\mathcal{J}^{aq} \equiv \frac{L^3}{iqu^3}\, \boldsymbol{t}_{a5} \,. \tag{A27}$$

### 3. Scalar Mode

The terms in Eq. (A21) constructed out of radial derivatives of $\mathcal{H}_{00}$, $\mathcal{H}_{aa}$, $\mathcal{H}_{qq}$ and $\mathcal{H}_{0q}$ are

$$\delta S^0_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4} \frac{L^3}{8\kappa_5^2 u^3} \Big[ 2\delta\mathcal{H}_{00}\, \partial_u \mathcal{H}_{00} - 4\delta\mathcal{H}_{0q}\, \partial_u \mathcal{H}_{0q} + 2\delta\mathcal{H}_{qq}\, \partial_u \mathcal{H}_{qq} - \delta\mathcal{H}_{aa}\, \partial_u \mathcal{H}_{aa} \Big] . \tag{A28}$$

Using the first order equations (4.33b)–(4.33d), this is seen to be

$$\delta S^0_{\rm B} = \lim_{u\to 0} \int \frac{d^4q}{(2\pi)^4} \Big[ \mathcal{A}_0\, \delta Z_0\, \partial_u Z_0 + \tfrac{1}{2}\delta\mathcal{H}_{00}\, \mathcal{J}^{00} + \tfrac{1}{2}\delta\mathcal{H}_{qq}\, \mathcal{J}^{qq} + \tfrac{1}{2}\delta\mathcal{H}_{aa}\, \mathcal{J}^{aa} + \delta\mathcal{H}_{0q}\, \mathcal{J}^{0q} \Big] , \tag{A29}$$

where

$$\mathcal{A}_0 = \frac{L^3}{6\kappa_5^2 u^3 (q^2{-}\omega^2)^2} \,, \tag{A30}$$

and

$$\mathcal{J}^{00} = \frac{iL^3}{3u^3(q^2{-}\omega^2)^2} \Big[ \omega \left(5q^2{-}3\omega^2\right) \boldsymbol{t}_{05} - q\left(q^2{-}3\omega^2\right) \boldsymbol{t}_{q5} + iuq^2\left(q^2{-}\omega^2\right) \boldsymbol{t}_{55}, \Big] , \tag{A31a}$$

$$\mathcal{J}^{qq} = \frac{iL^3}{3u^3(q^2{-}\omega^2)^2} \Big[ -\omega \left(\omega^2{-}3q^2\right) \boldsymbol{t}_{05} + q\left(5\omega^2{-}3q^2\right) \boldsymbol{t}_{q5} + iu\omega^2 (q^2{-}\omega^2)\, \boldsymbol{t}_{55} \Big] , \tag{A31b}$$

$$\mathcal{J}^{aa} = \frac{iL^3}{3u^3(q^2{-}\omega^2)} \Big[ \omega\, \boldsymbol{t}_{05} + q\, \boldsymbol{t}_{q5} - iu(q^2{-}\omega^2)\, \boldsymbol{t}_{55} \Big] , \tag{A31c}$$

$$\mathcal{J}^{0q} = \frac{2iL^3}{3u^3(q^2{-}\omega^2)^2} \Big[ -q\left(\omega^2{-}3q^2\right) \boldsymbol{t}_{05} - \omega\left(q^2{-}3\omega^2\right) \boldsymbol{t}_{q5} + iuq\left(q^2{-}\omega^2\right) \boldsymbol{t}_{55} \Big] . \tag{A31d}$$

## APPENDIX B: SMALL MOMENTUM ASYMPTOTICS

For very large or very small momentum (compared to $T$), one may find explicit asymptotic expressions for the homogeneous solutions $g_s^<$ and $g_s^>$, and derive the resulting asymptotic behavior of the SYM stress-energy tensor. In what follows we only consider the small momentum limit and, in particular, calculate $\mathcal{T}_s$ to $\mathcal{O}(1)$.

We write the homogeneous solutions $g_s^>$ and $g_s^<$ as a power series in $q$,

$$g_s^< = \phi_s^{(0)} + q\, \phi_s^{(1)} + q^2\, \phi_s^{(2)} + \mathcal{O}(q^3) \,, \tag{B1a}$$

$$g_s^> = \psi_s^{(0)} + q\, \psi_s^{(1)} + q^2\, \psi_s^{(2)} + \mathcal{O}(q^3) \,. \tag{B1b}$$

The differential equations (4.25), (4.30) and (4.34), without sources, are then solved order by order in $q$ subject to the boundary conditions discussed in Section IV E. In what follows we define $r \equiv \omega/q$.

### 1. Tensor Mode

For the tensor mode it is sufficient to expand the homogeneous solutions to Eq. (4.25) to $\mathcal{O}(q^0)$. Doing so one obtains

$$\phi_2^{(0)} = -u_h^4 \ln f, \tag{B2}$$

and

$$\psi_2^{(0)} = 1\,. \tag{B3}$$

The corresponding small momentum limit of the Wronskian is

$$W_2 = -\frac{4u^3}{f} + \mathcal{O}(q)\,. \tag{B4}$$

Substituting the above expansions into the definition of $\Delta \overleftrightarrow{Z}_2^{(4)}$ in Eq. (4.64) and performing the radial integral, one finds

$$\begin{aligned}\Delta \overleftrightarrow{Z}_2^{(4)} = &-\frac{\kappa_5^2\sqrt{\lambda}}{8\pi L^3 u_h\sqrt{1-v^2}}\,\frac{(vq_\perp)^2}{q^2}\,(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\\ &\times(\hat{\epsilon}_1\otimes\hat{\epsilon}_1-\hat{\epsilon}_2\otimes\hat{\epsilon}_2)+\mathcal{O}(q)\,.\end{aligned} \tag{B5}$$

Substituting $\Delta \overleftrightarrow{Z}_2^{(4)}$ into Eq. (4.54c) yields

$$\begin{aligned}\Delta \overleftrightarrow{\mathcal{T}}_2 = &-\frac{\sqrt{\lambda}}{4\pi u_h\sqrt{1-v^2}}\,\frac{(vq_\perp)^2}{q^2}\,(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\\ &\times(\hat{\epsilon}_1\otimes\hat{\epsilon}_1-\hat{\epsilon}_2\otimes\hat{\epsilon}_2)+\mathcal{O}(q)\,.\end{aligned} \tag{B6}$$

### 2. Vector Mode

For the vector mode it is necessary to expand the homogeneous solutions to Eq. (4.30) to $\mathcal{O}(q^2)$. Define for convenience

$$f_\pm \equiv 1 \pm \frac{u^2}{u_h^2}\,. \tag{B7}$$

The solutions $\phi_1^{(i)}$ and $\psi_1^{(i)}$ are given by

$$\phi_1^{(0)} = u^3\,, \tag{B8a}$$

$$\phi_1^{(1)} = 0\,, \tag{B8b}$$

$$\begin{aligned}\phi_1^{(2)} = &\frac{u^3u_h^2}{32}\Big(2\tanh^{-1}\frac{u^2}{u_h^2}+\ln\frac{f}{f_-^2}\Big)\\ &-\frac{r^2u^3u_h^2}{32}\Bigg(\frac{\pi^2}{3}+4\frac{u_h^2}{u^2}\ln f+8\tanh^{-1}\frac{u^2}{u_h^2}\\ &\quad-2\ln^2 2-\ln\frac{f_-}{16}\ln f_-+\ln f_+\ln\frac{f_+}{f_-^2}\\ &\quad-2\ln f\ln\frac{f_+}{f_-}-4\,\mathrm{Li}_2\frac{f_-}{2}\Bigg)\,,\end{aligned} \tag{B8c}$$

and

$$\psi_1^{(0)} = \frac{u^3}{u_h^3}\,, \tag{B9a}$$

$$\psi_1^{(1)} = \frac{iru^3}{2u_h^2}-\frac{iru}{4}\Big(2+\frac{u^2}{u_h^2}\ln\frac{f_-}{f_+}\Big)\,, \tag{B9b}$$

$$\begin{aligned}\psi_1^{(2)} = &\frac{uu_h}{32}\Big(4f_-+\frac{2u^2}{u_h^2}\tanh^{-1}\frac{u^2}{u_h^2}+\frac{u^2}{u_h^2}\ln\frac{f}{f_+^2}\Big)\\ &+\frac{r^2u^3}{32u_h}\Bigg(-4\frac{u_h^2}{u^2}\ln\frac{f}{4}-8\tanh^{-1}\frac{u^2}{u_h^2}\\ &\quad+\ln 2\ln 4-\ln\frac{16f^2}{f_-}\ln f_-+\ln 16\ln\frac{f_-}{f_+}\\ &\quad+2\ln(ff_-)\ln f_+-\ln^2 f_++4\,\mathrm{Li}_2\frac{f_-}{2}\Bigg)\,.\end{aligned} \tag{B9c}$$

The corresponding Wronskian is

$$W_1 = \frac{u^3}{f}\Big[irq-\frac{u_h}{4}\left(1+r^2\ln 4\right)q^2+\mathcal{O}(q^3)\Big]. \tag{B10}$$

Substituting the above solutions into Eq (4.67), one finds

$$\begin{aligned}\Delta \vec{Z}_1^{(3)} = &\frac{\kappa_5^2\sqrt{\lambda}}{\pi L^3 u_h\sqrt{1-v^2}}\,\frac{vq_\perp}{q}\Big[\frac{1}{irqu_h}-\frac{1-4r^2}{4r^2}+\mathcal{O}(q)\Big]\\ &\times(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\,\hat{\epsilon}_1\,.\end{aligned} \tag{B11}$$

From Eq. (4.54b) we therefore have

$$\begin{aligned}\Delta \vec{\mathcal{T}}_1 = &\frac{\sqrt{\lambda}}{2\pi u_h\sqrt{1-v^2}}\,\frac{vq_\perp}{q}\Big[\frac{i}{rqu_h}+\frac{1-4r^2}{4r^2}+\mathcal{O}(q)\Big]\\ &\times(2\pi)\delta(\omega-\boldsymbol{v}\cdot\boldsymbol{q})\,\hat{\epsilon}_1\,.\end{aligned} \tag{B12}$$

### 3. Scalar Mode

For the scalar mode it is sufficient to expand the homogeneous solutions to Eq. (4.34) to $\mathcal{O}(q)$. Doing so one obtains

$$\phi_0^{(0)} = \frac{4(2-3r^2)}{9(1-r^2)^2}\,u^4-\frac{(1-3r^2)[u^4+u_h^4(1-3r^2)]}{9(1-r^2)^2}\,\ln f\,, \tag{B13a}$$

$$\phi_0^{(1)} = 0\,, \tag{B13b}$$

and

$$\psi_0^{(0)} = \frac{u^4-(1-3r^2)u_h^4}{(2-3r^2)u_h^4}\,, \tag{B14a}$$

$$\begin{aligned}\psi_0^{(1)} = &\frac{ir}{4u_h^3(2-3r^2)}\Big\{(4+\ln 4)u^4-u_h^4\left(r^2\ln 64+4-\ln 4\right)\\ &-\left[u^4+\left(1-3r^2\right)u_h^4\right]\ln f\Big\}\,.\end{aligned} \tag{B14b}$$

The corresponding small momentum limit of the Wronskian is

$$W_0 = \frac{u^3[u^4 - 3(1-r^2)u_h^4]^2}{(1-r^2)^2(2-3r^2)f}\Bigg[-\frac{4(1-3r^3)}{9u_h^8} + \frac{2ir\left(r^2\ln 8 - \ln 2 + 2\right)}{9u_h^7}\,q + \mathcal{O}(q^2)\Bigg]. \tag{B15}$$

Substituting the above expansions into the definition of $\Delta Z_0^{(4)}$ in Eq. (4.70) and performing the radial integral, one finds

$$\Delta Z_0^{(4)} = \frac{\kappa_5^2\sqrt{\lambda}}{2\pi L^3\sqrt{1-v^2}}\frac{2-3r^2+v^2}{(1-3r^2)u_h}\,(2\pi)\delta(\omega - \boldsymbol{v}\cdot\boldsymbol{q}) \times\Bigg[-\frac{irq}{u_h} + \frac{9r^2(1-r^2)^2}{4(1-3r^2)} + \mathcal{O}(q^3)\Bigg]. \tag{B16}$$

Inserting $\Delta Z_0^{(4)}$ into Eq. (4.54a), we obtain

$$\Delta \mathcal{T}_0 = \frac{3\sqrt{\lambda}}{2\pi u_h\sqrt{1-v^2}}\,(2\pi)\delta(\omega - \boldsymbol{v}\cdot\boldsymbol{q}) \times\Bigg[\frac{-ir(1+v^2)}{(1-3r^2)u_h q} + \frac{r^2(2-3r^2+v^2)}{(1-3r^2)^2} + \mathcal{O}(q)\Bigg]. \tag{B17}$$


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